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NONCOMMUTATIVE INTERPOLATION AND POISSON TRANSFORMS

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ABSTRACT

General results of interpolation (e.g., Nevanlinna–Pick) by elements in the noncommutative analytic Toeplitz algebra F^∞ (resp., noncommutative disc algebra $\mathcal{A}_n$) with consequences to the interpolation by bounded operator-valued analytic functions in the unit ball of $\mathbb{C}^n$ are obtained. Noncommutative Poisson transforms are used to provide new von Neumann type inequalities. Completely isometric representations of the quotient algebra F^∞/J on Hilbert spaces, where J is any w^*-closed, 2-sided ideal of F^∞, are obtained and used to construct a w^*-continuous, F^∞/J-functional calculus associated to row contractions $T = [T_1, \ldots, T_n]$ when $f(T_1, \ldots, T_n) = 0$ for any $f \in J$. Other properties of the dual algebra F^∞/J are considered.

In [Po5], the second author proved the following version of von Neumann's inequality for row contractions: if $T_1, \ldots, T_n \in B(\mathcal{H})$ (the algebra of all bounded linear operators on the Hilbert space $\mathcal{H}$) and $T = [T_1, \ldots, T_n]$ is a contraction, i.e., $\sum_{i=1}^n T_i T_i^* \leq I_\mathcal{H}$, then for every polynomial $p(X_1, \ldots, X_n)$ on n noncommuting indeterminates,

$$(1) \qquad \|p(T_1, \ldots, T_n)\|_{B(\mathcal{H})} \leq \|p(S_1, \ldots, S_n)\|_{B(\mathcal{F}^2)},$$

where $S_1, \ldots, S_n$ are the left creation operators on the full Fock space $\mathcal{F}^2 = \mathcal{F}^2(\mathcal{H}_n)$ (we refer to Section 1 for notation and background material).

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As in [Po5], the noncommutative disc algebra $\mathcal{A}_n$ is the norm closed subalgebra in $B(\mathcal{F}^2)$ generated by $S_1, \ldots, S_n$ and the identity, and the Hardy (noncommutative analytic Toeplitz) algebra F^∞ is the WOT-closed algebra generated by $\mathcal{A}_n$ in $B(\mathcal{F}^2)$.

It was proved in [Po8] that if $T = [T_1, \ldots, T_n]$ is a contraction, then the map

$$\Phi_T: C^*(S_1, \ldots, S_n) \to B(\mathcal{H}); \quad \Phi_T(S_{i_1} \cdots S_{i_k} S_{j_1}^* \cdots S_{j_p}^*) = T_{i_1} \cdots T_{i_k} T_{j_1}^* \cdots T_{j_p}^*,$$

$1 \le i_1, \ldots, i_k, j_1, \ldots, j_p \le n$, is a completely contractive linear map, and $\Phi_T|_{\mathcal{A}_n}$ is a homomorphism. An elementary proof of this as well as an extension to a more general setting was obtained in [Po9], by the second author, using noncommutative Poisson transforms on C^*-algebras generated by isometries (we refer to Section 3 for a sketch of the proof).

Let J be a closed, 2-sided ideal of $\mathcal{A}_n$ with $J \subset \operatorname{Ker} \Phi_T$ and let $\mathcal{N}_J$ be the orthogonal of the image of J in $\mathcal{F}^2$. For each $i = 1, \ldots, n$, let $B_i := P_{\mathcal{N}_J} S_i|_{\mathcal{N}_J}$. Using noncommutative Poisson transforms [Po9], we will prove in Section 3 that, for a large class of row contractions $T = [T_1, \ldots, T_n]$ (including C_0-contractions), there is a unital, completely contractive, linear map $\Phi: C^*(B_1, \ldots, B_n) \to B(\mathcal{H})$ such that

$$\Phi(B_{i_1} \cdots B_{i_k} B_{j_1}^* \cdots B_{j_p}^*) = T_{i_1} \cdots T_{i_k} T_{j_1}^* \cdots T_{j_p}^*,$$

$1 \le i_1, \ldots, i_k, j_1, \ldots, j_p \le n$.

The noncommutative dilation theory for n-tuples of operators [Fr], [Bu], [Po1], [Po2] was used in [Po6] to obtain an F^∞-functional calculus associated to any completely non-coisometric contraction (in short c.n.c.) $T = [T_1, \ldots, T_n]$. More precisely, it was shown that the map $\Psi_T: F^\infty \to B(\mathcal{H})$ defined by

$$\Psi_T(f) = f(T_1, \ldots, T_n) := \text{SOT-}\lim_{r \to 1} f(rT_1, \ldots, rT_n)$$

is a WOT-continuous and completely contractive homomorphism. We will show that if J is a WOT-closed, 2-sided ideal of F^∞ with $J \subset \operatorname{Ker} \Psi_T$, then the map

$$p(B_1, \ldots, B_n) \mapsto p(T_1, \ldots, T_n)$$

can be extended to a WOT-continuous, completely contractive homomorphism from $\mathcal{W}(B_1, \ldots, B_n)$, the WOT-closed algebra generated by the compressions $B_i := P_{\mathcal{N}_J} S_i|_{\mathcal{N}_J}$, $i = 1, \ldots, n$, to $B(\mathcal{H})$.

Let us recall that F^∞ has $\mathbb{A}_1(1)$ property (see [DP1]), therefore the w^* and WOT topologies coincide on F^∞. An important step in proving the above-mentioned results is an extention of Sarason's result [S] to F^∞. More precisely, we will show that if J is a WOT-closed, 2-sided ideal of F^∞, then the map

$$\Psi: F^\infty/J \to B(\mathcal{N}_J), \quad \Psi(f + J) = P_{\mathcal{N}_J} f|_{\mathcal{N}_J}$$

is a w^*-continuous, completely isometric representation. In particular, for every $f \in F^\infty$,

$$\operatorname{dist}(f, J) = \|P_{\mathcal{N}_J} f | \mathcal{N}_J\|.$$

We present in this paper two proofs of this result: one is based on the noncommutative commutant lifting theorem [Po3] (see [SzF] and [FFr] for the classical case) and the characterization of the commutant of $S_1, \ldots, S_n$ from [Po7], and the other is based on noncommutative Poisson transforms [Po9] and representations of quotient algebras. This is a key result which leads to the noncommutative interpolation theorems of Caratheodory (obtained previously in [Po7]) and of Nevanlinna–Pick in the noncommutative analytic Toeplitz algebra F^∞. Let us mention just one consequence of our results to the interpolation by bounded analytic functions in the unit ball of $\mathbb{C}^n$. We will show that if $\lambda_1, \ldots \lambda_k$ are k distinct points in $\mathbb{B}_n$, the open unit ball of $\mathbb{C}^n$, $W_1, \ldots, W_k \in B(\mathcal{K})$ ($\mathcal{K}$ is a Hilbert space), and the operator matrix

$$\left[\frac{I_{\mathcal{K}} - W_j W_i^*}{1 - \langle \lambda_j, \lambda_i \rangle}\right]_{i,j=1,\ldots,k}$$

is positive definite, then there is an operator-valued analytic function $F\colon \mathbb{B}_n \to B(\mathcal{K})$ such that

$$\sup_{\zeta \in \mathbb{B}_n} \|F(\zeta)\| \leq 1 \quad \text{and} \quad F(\lambda_i) = W_j$$

for any $j = 1, \ldots, k$.

In fact, we obtain more general results of interpolation by elements in F^∞ (resp. $\mathcal{A}_n$) with consequences to the interpolation by bounded analytic functions in the unit ball of $\mathbb{C}^n$.

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After this paper was submitted for publication, we received a preprint from Davidson and Pitts [DP3], which has significant overlap with Section 2 of our paper. The principal overlapping parts are Theorem 2.4, Theorem 2.8, and Corollary 2.10. However, our proofs are quite different.

1. Notation and preliminary results

Unless explicitly stated, n stands for a cardinal number between $1 \leq n \leq \aleph_0$. Let $\mathcal{H}_n$ be an n-dimensional Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$. We consider the Full Fock space [E] of $\mathcal{H}_n$

$$\mathcal{F}^2 = \mathcal{F}^2(\mathcal{H}_n) = \bigoplus_{k \geq 0} \mathcal{H}_n^{\otimes k},$$

where $\mathcal{H}_n^{\otimes 0} = \mathbb{C}1$ and $\mathcal{H}_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of $\mathcal{H}_n$. We shall denote by $\mathcal{P}$ the set of all $p \in \mathcal{F}^2(\mathcal{H}_n)$ of the form

$$p = a_0 + \sum_{\substack{1 \le i_1, \dots, i_k \le n \\ 1 \le k \le m}} a_{i_1 \cdots i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}, \quad m \in \mathbb{N},$$

where $a_0, a_{i_1 \cdots i_k} \in \mathbb{C}$. The set $\mathcal{P}$ may be viewed as the algebra of the polynomials in n noncommuting indeterminates, with $p \otimes q$, $p, q \in \mathcal{P}$, as multiplication. For any bounded operators $T_1, \dots, T_n$ on a Hilbert space $\mathcal{H}$, define

$$p(T_1, \dots, T_n) := a_0 I_{\mathcal{H}} + \sum a_{i_1 \cdots i_k} T_{i_1} T_{i_2} \cdots T_{i_k}.$$

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \dots, g_n$ and the identity e. For each $\alpha \in \mathbb{F}_n^+$, define

$$e_\alpha := \begin{cases} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}, & \text{if } \alpha = g_{i_1} g_{i_2} \cdots g_{i_k}, \\ 1, & \text{if } \alpha = e. \end{cases}$$

It is easy to see that $\{e_\alpha : \alpha \in \mathbb{F}_n^+\}$ is an orthonormal basis of $\mathcal{F}^2$. We also use $\mathbb{F}_n^+$ to denote arbitrary products of operators. If $T_1, \dots, T_n \in B(\mathcal{H})$, define

$$T_\alpha := \begin{cases} T_{i_1} T_{i_2} \cdots T_{i_k}, & \text{if } \alpha = g_{i_1} g_{i_2} \cdots g_{i_k}, \\ I_{\mathcal{H}}, & \text{if } \alpha = e. \end{cases}$$

The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| = k$, if $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k}$, and $|\alpha| = 0$, if $\alpha = e$. For each $i = 1, \dots, n$, the left creation operator

$$S_i: \mathcal{F}^2 \to \mathcal{F}^2 \quad \text{is defined by} \quad S_i \psi = e_i \otimes \psi, \ \psi \in \mathcal{F}^2.$$

It is easy to see that $S_1, \dots, S_n$ are isometries with orthogonal ranges. As in [Po5], $\mathcal{A}_n$ is the norm closure of the algebra generated by $S_1, \dots, S_n$ and $I_{\mathcal{F}^2}$, and F^∞ is the weak operator topology closure of $\mathcal{A}_n$. Alternatively, we let $\mathcal{F}^\infty$ be the set of those $\varphi \in \mathcal{F}^2$ such that

$$\|\varphi\|_\infty := \sup\{\|\varphi \otimes p\|_2 : p \in \mathcal{P}, \|p\|_2 \le 1\} < \infty.$$

For $\varphi \in \mathcal{F}^\infty$, define $\varphi(S_1, \dots, S_n): \mathcal{F}^2 \to \mathcal{F}^2$ by $\varphi(S_1, \dots, S_n)\psi = \varphi \otimes \psi$. The norm $\|\varphi\|_\infty$ coincides with the operator norm of $\varphi(S_1, \dots, S_n)$. It will be useful later to view $\varphi \in \mathcal{F}^\infty$ as being an element in F^∞ and conversely. With this identification, $\mathcal{A}_n$ is the closure of $\mathcal{P}$ in the $\|\cdot\|_\infty$-norm.

We recall from [Po7] the characterization of the commutant of $S_1, \dots, S_n$. Define the flipping operator $U\colon \mathcal{F}^2 \to \mathcal{F}^2$ by

$$U(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = e_{i_k} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1},$$

and let $\tilde{\varphi} := U\varphi$. It is easy to see that U is a unitary operator, which satisfies $U(\varphi \otimes \psi) = \tilde{\psi} \otimes \tilde{\varphi}$, and $U^2 = I$. An operator $A \in B(\mathcal{F}^2)$ commutes with $S_1, \dots, S_n$ if and only if there exists $\phi \in \mathcal{F}^\infty$ such that $Ah = h \otimes \tilde{\phi}, \quad h \in \mathcal{F}^2$. Notice that $A = U^* \phi(S_1, \dots, S_n) U$.

In [Po2], the second author defined $\varphi \in \mathcal{F}^\infty$ to be inner if $\varphi(S_1, \dots, S_n)$ is an isometry, and outer if $\varphi(S_1, \dots, S_n)$ has dense range. A complete description of the invariant subspace structure of $\mathcal{F}^\infty$ was obtained in [Po2] (even in a more general setting), using a noncommutative version of the Wold decomposition (see [Po1]). A family of inner operators $\{\varphi_i\colon i \in I\}$ is called orthogonal, if whenever $i \neq j$, $\mathcal{F}^2 \otimes \tilde{\varphi}_i$ is orthogonal to $\mathcal{F}^2 \otimes \tilde{\varphi}_j$; or equivalently, $\varphi_i \otimes \mathcal{F}^2$ is orthogonal to $\varphi_j \otimes \mathcal{F}^2$. It follows from [Po2; Theorem 2.2] that a subspace $\mathcal{M}$ of $\mathcal{F}^2$ is invariant under $S_1, \dots, S_n$ if and only if $\mathcal{M} = \bigoplus_{i \in I} \mathcal{F}^2 \otimes \tilde{\varphi}_i$, for some family of orthogonal inner operators.

The second author obtained in [Po4] an inner-outer factorization which implies that any $\eta \in \mathcal{F}^\infty$ can be factored as $\eta = \varphi \otimes \psi$, where φ is inner and ψ is outer. The same factorization result was proved for elements of $\mathcal{F}^2$ in [APo], where $\eta \in \mathcal{F}^2$ was said to be outer if there exists a sequence of polynomials $p_n \in \mathcal{P}$ such that $\psi \otimes p_n \to 1$ in the norm of $\mathcal{F}^2$ (this last result was also obtained recently in [DP1]). Let us mention that we proved in [APo] that the noncommutative analytic Toeplitz algebra in n noncommuting variables $\mathcal{F}^\infty$ is reflexive. Recently, Davidson and Pitts [DP1] proved that this algebra is hyper-reflexive. They also studied in [DP2] the algebraic structure of $\mathcal{F}^\infty$ (in their notation $\mathcal{L}_n$).

Now let us recall some general facts about duality in Banach spaces. Let X be a Banach space with predual X_* and dual X^*, and let $\mathcal{S} \subset X$. The preannihilator of $\mathcal{S}$ in X_* is the set ${}^\perp\mathcal{S} = \{f \in X_*\colon \langle f, x\rangle = 0 \text{ for all } x \in \mathcal{S}\}$, and the annihilator of $\mathcal{S}$ in X^* is the set $\mathcal{S}^\perp = \{f \in X^*\colon \langle x, f\rangle = 0 \text{ for all } x \in \mathcal{S}\}$. If $\mathcal{S}$ is w^*-closed, it is well known that $({}^\perp S)^* = X/S$, $({}^\perp\mathcal{S})^\perp = \mathcal{S}$, $(X_*/{}^\perp\mathcal{S})^* = \mathcal{S}$, and $({}^\perp\mathcal{S})^{**} = \mathcal{S}^\perp$ (see [Ru]).

The predual of $B(\mathcal{H})$ is the space of trace class operators $c_1(\mathcal{H})$, under the trace duality. That is, if $T \in c_1(\mathcal{H})$ and $A \in B(\mathcal{H})$, then $\langle T, A\rangle := tr(TA)$. A w^*-closed subspace $\mathcal{S}$ of $B(\mathcal{H})$ has property $\mathbb{A}_1$ if for every w^*-continuous linear map $f\colon \mathcal{S} \to \mathbb{C}$, there exists $h, k \in \mathcal{H}$ such that for all $T \in \mathcal{S}$, $f(T) = \langle Th, k\rangle$. Moreover, if for every $\epsilon > 0$, h and k can be chosen so that $\|h\|\|h\| \leq (1+\epsilon)\|f\|$, $\mathcal{S}$

has property $\mathbb{A}_1(1)$ (we refer to [BFP] for more information). If $n \geq 2$, Davidson and Pitts [DP1] proved that $F^\infty \subset B(\mathcal{F}^2)$ has property $\mathbb{A}_1(1)$, and Bercovici [B] proved that $M_k(F^\infty)$ has property $\mathbb{A}_1(1)$ for each $k \geq 1$ (i.e., F^∞ has property $\mathbb{A}_{\aleph_0}(1)$).

We refer to [Ar1], [P], and [Pi] for results on completely bounded maps and operator spaces.

Let J be a WOT-closed, 2-sided ideal of F^∞ and define

$$\mathcal{M}_J = \overline{\{\varphi \otimes \psi : \varphi \in J, \psi \in \mathcal{F}^2\}}^{\|\cdot\|_2}, \text{ and}$$
$$\mathcal{N}_J = \mathcal{F}^2 \ominus \mathcal{M}_J.$$

Recall from [S] that a subspace $\mathcal{N} \subset \mathcal{H}$ is semi-invariant under a semigroup of operators $\Sigma \subset B(\mathcal{H})$ if for every $T_1, T_2 \in \Sigma$, $P_\mathcal{N} T_1 P_\mathcal{N} T_2 P_\mathcal{N} = P_\mathcal{N} T_1 T_2 P_\mathcal{N}$. It is well known that if $\mathcal{N}_1, \mathcal{N}_2$ are invariant subspaces under Σ and $\mathcal{N}_2 \subset \mathcal{N}_1$ then $\mathcal{N}_1 \ominus \mathcal{N}_2$ is semi-invariant under Σ.

LEMMA 1.1: *If J is a* WOT*-closed, 2-sided ideal of F^∞, then the subspaces $\mathcal{N}_J$ and $U\mathcal{N}_J$ are invariant under each S_i^*, $i = 1, 2, \ldots, n$.*

Proof: Since J is a left ideal, $\mathcal{M}_J$ is invariant under $S_1, \ldots, S_n$ and, hence, $\mathcal{N}_J$ is invariant under $S_1^*, \ldots, S_n^*$. Moreover, since J is a right ideal, the set $\{\varphi : \varphi(S_1, \ldots, S_n) \in J\}$ is dense in $\mathcal{M}_J$. Similarly, one can prove that $U\mathcal{N}_J$ is invariant to each S_i^*, $i = 1, 2, \ldots, n$. ∎

PROPOSITION 1.2: *Let J be a w^*-closed, 2-sided ideal of F^∞ and $g \in {}^\perp J \subset (F^\infty)_*$. For each $\epsilon > 0$, there exist $\psi_1, \psi_2 \in \mathcal{N}_J$ satisfying $\|\psi_1\|_2 \|\psi_2\|_2 \leq (1+\epsilon)\|g\|$ such that for every $\eta \in F^\infty$, $g(\eta) = \langle \eta \otimes \psi_1, \psi_2 \rangle$. Conversely, if $\psi_1, \psi_2 \in \mathcal{N}_J$ and $g(\eta) := \langle \eta \otimes \psi_1, \psi_2 \rangle$ for any $\eta \in F^\infty$, then $g \in {}^\perp J$.*

Proof: Let $g \in {}^\perp J \subset (F^\infty)_*$ and let $\epsilon > 0$. Since F^∞ has the $\mathbb{A}_1(1)$ property, find $\varphi_1, \varphi_2 \in \mathcal{F}^2$ satisfying $\|\varphi_1\|_2 \|\varphi_2\|_2 \leq (1+\epsilon)\|g\|$ such that for every $\xi \in F^\infty$, $\langle g, \xi \rangle = \langle \xi \otimes \varphi_1, \varphi_2 \rangle$. Factor $\varphi_1 = \eta_1 \otimes \eta_2$, where $\tilde{\eta}_1$ is outer and $\tilde{\eta}_2$ is inner. Hence, there exist a sequence of polynomials $p_n \in \mathcal{P}$ such that $p_n \otimes \eta_1 \to 1$ in the norm of $\mathcal{F}^2$, and $\mathcal{F}^2 \otimes \eta_2$ is a closed subspace of $\mathcal{F}^2$. Let P_{η_2} be the orthogonal projection onto $\mathcal{F}^2 \otimes \eta_2$, and write $P_{\eta_2}\varphi_2 = \psi_2 \otimes \eta_2$ for some $\psi_2 \in \mathcal{F}^2$. Then for each $\xi \in \mathcal{F}^\infty$,

$$\begin{aligned} g(\xi) &= \langle \xi \otimes \varphi_1, \varphi_2 \rangle = \langle \xi \otimes \eta_1 \otimes \eta_2, \varphi_2 \rangle = \langle P_{\eta_2}(\xi \otimes \eta_1 \otimes \eta_2), \varphi_2 \rangle \\ &= \langle \xi \otimes \eta_1 \otimes \eta_2, P_{\eta_2}(\varphi_2) \rangle = \langle \xi \otimes \eta_1 \otimes \eta_2, \psi_2 \otimes \eta_2 \rangle = \langle \xi \otimes \eta_1, \psi_2 \rangle. \end{aligned}$$

The last equality follows because the operator on $\mathcal{F}^2$ that multiplies from the right by η_2 is an isometry (this is the equivalent to $\tilde{\eta}_2$ inner).

We will show that $\psi_2 \in \mathcal{N}_J$. Let $\xi \in J$. Since J is a right ideal, $\xi \otimes p_n \in J$ for each n. Hence, $0 = g(\xi \otimes p_n) = \langle \xi \otimes p_n \otimes \eta_1, \psi_2 \rangle \to \langle \xi, \psi_2 \rangle$. Therefore, $\langle \xi, \psi_2 \rangle = 0$. Since $\{\varphi \in \mathcal{F}^2 \colon \varphi(S_1, \ldots, S_n) \in J\}$ is dense in $\mathcal{M}_J$ we conclude that $\psi_2 \in \mathcal{N}_J$.

Recall that $\mathcal{N}_J$ is invariant under S_α^* and let $\psi_1 = P_{\mathcal{N}_J}(\eta_1)$. For each $\alpha \in \mathbb{F}_n^+$,

$$\begin{aligned} g(e_\alpha) &= \langle e_\alpha \otimes \eta_1, \psi_2 \rangle = \langle \eta_1, S_\alpha^*(\psi_2) \rangle \\ &= \langle \eta_1, P_{\mathcal{N}_J} S_\alpha^*(\psi_2) \rangle = \langle P_{\mathcal{N}_J}(\eta_1), S_\alpha^*(\psi_2) \rangle \\ &= \langle \psi_1, S_\alpha^*(\psi_2) \rangle = \langle e_\alpha \otimes \psi_1, \psi_2 \rangle. \end{aligned}$$

The converse is straightforward. This completes the proof. ∎

As a consequence of Proposition 1.2, we obtain the following.

Proposition 1.3: *For every* $\varphi \in F^\infty, \operatorname{dist}(\varphi, J) = \|P_{\mathcal{N}_J} \varphi(S_1, \ldots, S_n)|_{\mathcal{N}_J}\|$. *Consequently, the map* $\Phi \colon F^\infty / J \to B(\mathcal{N}_J)$ *defined by*

$$\Phi(\varphi + J) = P_{\mathcal{N}_J} \varphi(S_1, \ldots, S_n)|_{\mathcal{N}_J}$$

is an isometric homomorphism.

Proof: If $\psi \in J$ it is clear that $P_{\mathcal{N}_J} \psi(S_1, \ldots, S_n)|_{\mathcal{N}_J} = 0$. Hence, for every $\varphi \in F^\infty$, $\|P_{\mathcal{N}_J} \varphi(S_1, \ldots, S_n)|_{\mathcal{N}_J}\| \le \operatorname{dist}(\varphi, J)$.

Suppose now that $\varphi \notin J$. Since $({}^\perp J)^* = F^\infty / J$, for every $\epsilon > 0$ there exists $f \in {}^\perp J$, $\|f\| < 1$ such that $|f(\varphi)| > \operatorname{dist}(\varphi, J) - \epsilon$. By Proposition 1.2, there exist $\xi_1, \xi_2 \in \mathcal{N}_J$ such that $\|\xi_1\|_2 \|\xi_2\|_2 \le 1$ and $f(\varphi) = \langle \varphi(S_1, \ldots, S_n)\xi_1, \xi_2 \rangle$. Hence, $\|P_{\mathcal{N}_J} \varphi(S_1, \ldots, S_n)|_{\mathcal{N}_J}\| \ge \operatorname{dist}(\varphi, J) - \epsilon$. Since $\epsilon > 0$ is arbitrary, we finish the proof. ∎

It should be noted that Proposition 1.3 is all one really needs to obtain the scalar version of Carathéodory or Nevanlinna–Pick interpolation in F^∞.

We know from [Po5], [Po6] that the set $\mathcal{P}$ of all polynomials in $S_1, \ldots, S_n$ is WOT-dense in $\mathcal{F}^\infty$. Indeed, if $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha$ is in $\mathcal{F}^\infty$ and

$$f_r := \sum_{\alpha \in \mathbb{F}_n^+} r^{|\alpha|} a_\alpha e_\alpha$$

for any $0 < r < 1$ then $\text{SOT-lim}_{r \to 1} f_r = f$ and $\|f_r\|_\infty \le \|f\|_\infty$ (see [Po6]). On the other hand, $f_r \in \mathcal{A}_n$. Indeed, since $\|\sum_{|\alpha|=k} a_\alpha S_\alpha\| = (\sum_{|\alpha|=k} |a_\alpha|^2)^{1/2}$, we have

$$\begin{aligned} \sum_{k=0}^\infty r^k \|\sum_{|\alpha|=k} a_\alpha S_\alpha\| &= \sum_{k=0}^\infty r^k (\sum_{|\alpha|=k} |a_\alpha|^2)^{1/2} \\ &\le (\sum_{k=0}^\infty r^k) \|f\|_2. \end{aligned}$$

Therefore, $\sum_{\alpha\in\mathbb{F}_n^+} r^{|\alpha|}a_\alpha S_\alpha$ converges in norm, so that $f_r \in \mathcal{A}_n$. Taking into account that $\mathcal{P}$ is norm dense in $\mathcal{A}_n$, the result follows.

We will use the same notation as above if J is a closed, 2-sided ideal in $\mathcal{A}_n$.

LEMMA 1.4: *Let $J \subset \mathcal{A}_n$ be a 2-sided ideal of $\mathcal{A}_n$ and let J_w be the* WOT*-closed, 2-sided ideal generated by J in F^∞. Then $J_w = \bar{J}^{WOT}$ and $\mathcal{N}_{J_w} = \mathcal{N}_J$.*

Proof: We need to show that $\bar{J}^{\mathrm{WOT}}$ is a 2-sided ideal in F^∞. Consider $\psi, \phi \in F^\infty$, $f \in \bar{J}^{WOT}$, and let $\{g_i\} \subset J$ be a net WOT-convergent to f.

Since J is a 2-sided ideal of $\mathcal{A}_n$, $\phi_r g_i \psi_{r'} \in J$ for any $r, r' \in (0,1)$ and any i. Using the remarks preceding this lemma, it is easy to see, by taking appropriate limits, that $\phi f \psi \in \bar{J}^{WOT}$. Now let us show that $\mathcal{N}_{J_w} = \mathcal{N}_J$. Since $J \subset J_w$, it is clear that $\mathcal{N}_J \supset \mathcal{N}_{J_w}$. Let $f \in \bar{J}^{WOT}$, $\psi \in \mathcal{F}^2$, and choose $\{g_i\} \subset J$ such that WOT-$\lim_i g_i = f$. If $x \in \mathcal{N}_J$ we have

$$\langle x, f \otimes \psi\rangle = \lim_i \langle x, g_i(S_1, \ldots, S_n)\psi\rangle = 0.$$

Therefore, $x \in \mathcal{N}_{J_w}$, which proves that $\mathcal{N}_J \subset \mathcal{N}_{J_w}$. This completes the proof. ∎

2. Non-commutative interpolation in F^∞

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and I be a set of indices with $\dim \mathcal{K} = \operatorname{card} I = \gamma$. Denote $\oplus_\gamma \mathcal{H} := \oplus_{i\in I}\mathcal{H}_i$ where $\mathcal{H}_i := \mathcal{H}$, and notice that, under the canonical identification $\oplus_\gamma \mathcal{H} = \mathcal{H} \otimes \mathcal{K}$ (Hilbert tensor product), each operator $X \in B(\mathcal{H} \otimes \mathcal{K})$ can be seen as a *matrix* of operators in $B(\mathcal{H})$, i.e., $X = [X_{\alpha\beta}]_{\alpha,\beta\in I}$ with $X_{\alpha\beta} \in B(\mathcal{H})$. For any $\mathcal{U} \subset B(\mathcal{H})$, we denote

$$M_\gamma(\mathcal{U}) = \{[u_{\alpha\beta}] \in B(\oplus_\gamma \mathcal{H}) \colon u_{\alpha\beta} \in \mathcal{U}; \alpha, \beta \in I\}.$$

It is clear now that $\{u \otimes I_\mathcal{K};\ u \in \mathcal{U}\}' = M_\gamma(\mathcal{U}')$ (where $'$ stands for commutant).

Let us recall from [Po1] that $T = [T_1, \ldots, T_n]$ is called C_0-contraction if T is a contraction and

$$\text{SOT-}\lim_{k\to\infty} \sum_{\alpha\in\mathbb{F}_n^+, |\alpha|=k} T_\alpha T_\alpha^* = 0. \tag{2.1}$$

For example, if $\sum_{i=1}^n T_i T_i^* \le \rho I_\mathcal{H}$ for some $0 < \rho < 1$, then $[T_1, \ldots, T_n]$ is a C_0-contraction.

The following result is an extension of Sarason's result [S] and a consequence of the noncommutative commutant lifting theorem [Po3] and the characterization of the commutant of $\{S_1, \ldots, S_n\}$ from [Po7].

THEOREM 2.1: *Let $\mathcal{K}$ be a Hilbert space with $\dim\mathcal{K} = \gamma$ and let $\mathcal{N} \subset \mathcal{F}^2$ be an invariant subspace for $S_1^*, \dots, S_n^*$. If $T \in B(\mathcal{N} \otimes \mathcal{K})$ commutes with each $X_i := P_\mathcal{N} S_i|_\mathcal{N} \otimes I_\mathcal{K}$, $i = 1, 2, \dots, n$, then there is $\Phi(S_1, \dots, S_n) := [\phi_{\alpha,\beta}(S_1, \dots, S_n)]$ in $M_\gamma(F^\infty)$ such that $\|\Phi(S_1, \dots, S_n)\| = \|T\|$ and*

$$P_{\mathcal{N}\otimes\mathcal{K}}[U^*\phi_{\alpha\beta}(S_1, \dots, S_n)U] = TP_{\mathcal{N}\otimes\mathcal{K}},$$

where $P_{\mathcal{N}\otimes\mathcal{K}}$ is the orthogonal projection of $\mathcal{F}^2 \otimes \mathcal{K}$ onto $\mathcal{N} \otimes \mathcal{K}$, and U is the flipping operator on $\mathcal{F}^2$.

Proof: It is clear that $[P_\mathcal{N} S_1|_\mathcal{N}, \dots, P_\mathcal{N} S_n|_\mathcal{N}]$ is a C_0-contraction and, according to [Po1], its minimal isometric dilation is $[S_1, \dots, S_n]$. Therefore, the minimal isometric dilation of $[X_1, \dots, X_n]$ is $[S_1 \otimes I_\mathcal{K}, \dots, S_n \otimes I_\mathcal{K}]$. According to the noncommutative commutant lifting theorem [Po3], there is $A \in \{S_i \otimes I_\mathcal{K}; i = 1, 2, \dots, n\}'$ such that $\|A\| = \|T\|$ and $A^*|_{\mathcal{N}\otimes\mathcal{K}} = T^*$. Therefore, there exists $a_{\alpha,\beta} \in \{S_1, \dots, S_n\}'$ such that $A = [a_{\alpha\beta}] \in M_\gamma(\{S_1, \dots, S_n\}') \subset B(\oplus_\gamma \mathcal{F}^2)$. Using the characterization of the commutant of $\{S_1, \dots, S_n\}$ from [Po7], we find $\phi_{\alpha\beta} \in \mathcal{F}^\infty$ such that $a_{\alpha\beta} = U^*\phi_{\alpha\beta}(S_1, \dots, S_n)U$, where U is the flipping operator on $\mathcal{F}^2$. Therefore

$$A = [U^*\phi_{\alpha\beta}(S_1, \dots, S_n)U] \text{ and } P_{\mathcal{N}\otimes\mathcal{K}}[U^*\phi_{\alpha\beta}(S_1, \dots, S_n)U] = TP_{\mathcal{N}\otimes\mathcal{K}}. \quad \blacksquare$$

Notice that if $n = 1$ we find again Sarason's result [S].

LEMMA 2.2: *Let $T_i \in B(\mathcal{H})$ be such that $T := [T_1, \dots, T_n]$ is a C_0-contraction and let $f_{\alpha\beta}(S_1, \dots, S_n) \in F^\infty, \alpha, \beta \in I$, be such that $[f_{\alpha\beta}(S_1, \dots, S_n)]_{\alpha,\beta\in I}$ is in $B(\oplus_\gamma \mathcal{F}^2)$ $(\gamma = \operatorname{card} I)$. Then $[f_{\alpha\beta}(T_1, \dots, T_n)] \in B(\oplus_\gamma \mathcal{H})$ and*

$$\|[f_{\alpha\beta}(T_1, \dots, T_n)]\| \le \|[f_{\alpha\beta}(S_1, \dots, S_n)]\|.$$

Proof: According to [Po1], the minimal isometric dilation of $T = [T_1, \dots, T_n]$ is $[S_1 \otimes I_\mathcal{L}, \dots, S_n \otimes I_\mathcal{L}]$ for some Hilbert space $\mathcal{L}$. According to Theorem 3.6 from [Po5], for any $f_{\alpha\beta} \in \mathcal{F}^\infty$, $f_{\alpha\beta}(T_1, \dots, T_n) = P_\mathcal{H}(f_{\alpha\beta}(S_1, \dots, S_n) \otimes I_\mathcal{L})|_\mathcal{H}$. The rest of the proof is straightforward. $\blacksquare$

If $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$ is such that $|\zeta| := (|\zeta_1|^2 + \dots + \zeta_n|^2)^{1/2} < 1$, and $f(S_1, \dots, S_n) \in F^\infty$, then, according to the F^∞-functional calculus [Po6], we infer that $|f(\zeta_1, \dots, \zeta_n)| \le \|f(S_1, \dots, S_n)\|$. Therefore, $f(\zeta_1, \dots, \zeta_n)$ is an analytic function in $\mathbb{B}_n$. Moreover, we deduce the following.

COROLLARY 2.3: *If* $f_{\alpha\beta} \in F^\infty$, $\alpha, \beta \in I$, card $I = \gamma$, *and* $[f_{\alpha\beta}(S_1, \ldots, S_n)]_{\alpha,\beta\in I}$ *is in* $B(\oplus_\gamma \mathcal{F}^2)$, *then* $\Phi(\zeta) := [f_{\alpha\beta}(\zeta)]$ *is an operator-valued analytic function in* $\mathbb{B}_n$. *Moreover,* $\sup_{\zeta\in\mathbb{B}_n} \|\Phi(\zeta)\| \le \|[f_{\alpha\beta}(S_1, \ldots, S_n)]\|$.

A consequence of Theorem 2.1 is the following extension of the Nevanlinna–Pick problem to the noncommutative Toeplitz algebra F^∞.

THEOREM 2.4: *Let* $\lambda_1, \ldots, \lambda_k$ *be* k *distinct points in* $\mathbb{B}_n$ *and let* $W_1, \ldots, W_k$ *be in* $B(\mathcal{K})$, *where* $\mathcal{K}$ *is a Hilbert space with* $\dim \mathcal{K} = \gamma$. *Then there exists* $\Phi(S_1, \ldots, S_n) := [\phi_{\alpha,\beta}(S_1, \ldots, S_n)]$ *in* $M_\gamma(F^\infty)$, *such that* $\|\Phi(S_1, \ldots, S_n)\| \le 1$ *and* $\Phi(\lambda_j) = W_j$, $j = 1, 2, \ldots, k$, *if and only if the operator matrix*

$$\left[\frac{I_{\mathcal{K}} - W_j W_i^*}{1 - \langle \lambda_j, \lambda_i \rangle}\right]_{i,j=1,2,\ldots,k} \tag{2.2}$$

is positive definite.

Proof: For each $i = 1, \ldots, k$, let $\lambda_i := (\lambda_{i1}, \ldots, \lambda_{in}) \in \mathbb{C}^n$ and, for $\alpha = g_{j_1} g_{j_2} \cdots g_{j_m}$ in $\mathbb{F}_n^+$, let $\lambda_{i\alpha} := \lambda_{ij_1} \lambda_{ij_2} \ldots \lambda_{ij_m}$ and $\lambda_e = 1$. Define

$$z_{\lambda_i} := \sum_{\alpha\in\mathbb{F}_n^+} \overline{\lambda}_{i\alpha} e_\alpha, \quad i = 1, 2, \ldots, n,$$

and notice that, for any $\phi = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha e_\alpha$ in $\mathcal{F}^2$, $\langle \phi, z_{\lambda_i} \rangle = \phi(\lambda_i)$.

If $\phi \in \mathcal{F}^\infty$ then $\langle \phi, z_{\lambda_i} \rangle = \langle 1, \phi(S_1, \ldots, S_n)^* z_{\lambda_i} \rangle$, where $S_1, \ldots, S_n$ are the left creation operators on the full Fock space $\mathcal{F}^2$. It is clear that $S_i^* z_{\lambda_j} = \overline{\lambda}_{ji} z_{\lambda_j}$ for any $i = 1, \ldots, n;\ j = 1, \ldots, k$. Denote

$$\mathcal{N} := \operatorname{span}\{z_{\lambda_j} :\ j = 1, \ldots, k\}$$

and define $X_i \in B(\mathcal{N} \otimes \mathcal{K})$ by $X_i = P_{\mathcal{N}} S_i|_{\mathcal{N}} \otimes I_{\mathcal{K}}$. Since $z_{\lambda_1}, \ldots, z_{\lambda_k}$ are linearly independent, we can define $T \in B(\mathcal{N} \otimes \mathcal{K})$ by setting

$$T^*(z_{\lambda_j} \otimes h) = z_{\lambda_j} \otimes W_j^* h \tag{2.3}$$

for any $h \in \mathcal{K}$, $j = 1, \ldots, k$. Notice that for each $i = 1, \ldots, k$, $TX_i = X_i T$. Indeed,

$$\begin{aligned} X_i^* T^*(z_{\lambda_j} \otimes h) &= X_i^*(z_{\lambda_j} \otimes W_j^* h) = S_i^* z_{\lambda_j} \otimes W_j^* h \\ &= \overline{\lambda}_{ji} z_{\lambda_j} \otimes W_j^* h \end{aligned}$$

and

$$T^* X_i^*(z_{\lambda_j} \otimes h) = T^*(\overline{\lambda}_{ji} z_{\lambda_j} \otimes h) = \overline{\lambda}_{ji} z_{\lambda_j} \otimes W_j^* h.$$

Since $\mathcal{N}$ is invariant under S_i^*, $i = 1, \ldots, n$, according to Theorem 2.1, there exists $[\phi_{\alpha,\beta}(S_1, \ldots, S_n)] \in M_\gamma(F^\infty)$ such that $P_{\mathcal{N}\otimes\mathcal{K}}[U^*\phi_{\alpha,\beta}(S_1, \ldots, S_n)U] = TP_{\mathcal{N}\otimes\mathcal{K}}$, and

$$\| [\phi_{\alpha,\beta}(S_1, \ldots, S_n)] \| = \|T\|. \tag{2.4}$$

Let us show that $\Phi(S_1, \ldots, S_n) := [\phi_{\alpha,\beta}(S_1, \ldots, S_n)]$ satisfies $\Phi(\lambda_j) = W_j$ for any $j = 1, \ldots, k$, if and only if

$$[P_{\mathcal{N}} U^* \phi_{\alpha,\beta}(S_1, \ldots, S_n) U|_{\mathcal{N}}] = T. \tag{2.5}$$

To prove this, notice first that $U(z_{\lambda_j}) = z_{\lambda_j}$, $j = 1, \ldots, k$, and

$$\langle \phi_{\alpha,\beta}(S_1, \ldots, S_n) z_{\lambda_j}, z_{\lambda_j} \rangle = \phi_{\alpha,\beta}(\lambda_j) \langle z_{\lambda_j}, z_{\lambda_j} \rangle.$$

Due to these relations, (2.3), and (2.5), it is easy to see that, for any $j = 1, \ldots, k$ and $h, h' \in \mathcal{K}$, we have

$$\begin{aligned} \langle [U^*\phi_{\alpha,\beta}(S_1, \ldots, S_n)U](z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle &= \langle z_{\lambda_j}, z_{\lambda_j} \rangle \langle \Phi(\lambda_j) h, h' \rangle \\ &= \langle T(z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle = \langle z_{\lambda_j} \otimes h, z_{\lambda_j} \otimes W_j^* h' \rangle \\ &= \langle z_{\lambda_j}, z_{\lambda_j} \rangle \langle W_j h, h' \rangle. \end{aligned}$$

Now, it is clear that $\Phi(\lambda_j) = W_j$ for any $j = 1, \ldots, k$ if and only if (2.5) holds. On the other hand, (2.4) shows that $\|\Phi(S_1, \ldots, S_n)\| \leq 1$ if and only if $\|T\| \leq 1$. The latter condition is equivalent to

$$\langle g, g \rangle - \langle T^* g, T^* g \rangle \geq 0$$

for any $g = \sum_{i=1}^k z_{\lambda_j} \otimes h_j$ in $\mathcal{N} \otimes \mathcal{K}$. This inequality is equivalent to

$$\sum_{i,j=1}^k \langle z_{\lambda_i}, z_{\lambda_j} \rangle \langle (I - W_j W_i^*) h_i, h_j \rangle \geq 0, \tag{2.6}$$

for any $h_j \in \mathcal{K}$. Since

$$\begin{aligned} \langle z_{\lambda_i}, z_{\lambda_j} \rangle = z_{\lambda_i}(\lambda_j) &= \sum_{\alpha \in \mathbb{F}_n^+} \overline{\lambda}_{i\alpha} \lambda_{j\alpha} \\ &= 1 + \langle \lambda_j, \lambda_i \rangle + \langle \lambda_j, \lambda_i \rangle^2 + \cdots \\ &= \frac{1}{1 - \langle \lambda_j, \lambda_i \rangle}, \end{aligned}$$

inequality (2.6) holds if and only if the matrix (2.2) is positive definite. This completes the proof. ∎

COROLLARY 2.5: *If $n = 1$ we find again the Nevanlinna–Pick interpolation theorem* (see [Pic], [N]).

Notice that the proof of Theorem 2.4 works also for arbitrary families $\{\lambda_j\}_{j\in J}$ of distinct elements in $\mathbb{B}_n$, the open unit ball of $\mathbb{C}^n$.

THEOREM 2.6: *Let $\{\lambda_j\}_{j\in J}$ be distinct elements in $\mathbb{B}_n$ and let $\{W_j\}_{j\in J} \subset B(\mathcal{K})$, where $\mathcal{K}$ is a Hilbert space of dimension γ. Then there exists $\Phi(S_1, \ldots, S_n)$ in $M_\gamma(F^\infty)$ such that $\|\Phi(S_1, \ldots, S_n)\| \le 1$ and $\Phi(\lambda_j) = W_j$ for all $j \in J$ if and only if*

$$\sum_{i,j\in J} \left\langle \frac{I_{\mathcal{K}} - W_j W_i^*}{1 - \langle \lambda_j, \lambda_i \rangle} h_i, h_j \right\rangle \ge 0$$

for any $\{h_j\}_{j\in J} \subset \mathcal{K}$ such that $\{j : h_j \neq 0\}$ is finite.

Combining Theorem 2.4 with Corollary 2.3, we obtain the following sufficient condition for interpolation in the open unit ball of $\mathbb{C}^n$.

COROLLARY 2.7: *Let $\lambda_1, \ldots \lambda_k$ be k distinct points in $\mathbb{B}_n$ and let $W_1, \ldots, W_k$ be in $B(\mathcal{K})$. If the matrix*

$$\left[\frac{1 - W_j W_i^*}{1 - \langle \lambda_j, \lambda_i \rangle}\right]_{i,j=1,\ldots,k} \tag{2.7}$$

is positive definite, then there is an operator-valued analytic function $F\colon \mathbb{B}_n \to B(\mathcal{K})$ such that

$$\sup_{\zeta\in\mathbb{B}_n} \|F(\zeta)\| \le 1 \quad \text{and} \quad F(\lambda_i) = W_j$$

for any $j = 1, \ldots, k$.

Arveson [Ar2] showed that there are functions F in $H^\infty(\mathbb{B}_n)$ for which there are no $f \in F^\infty$ such that $f(\lambda) = F(\lambda)$ for each $\lambda \in \mathbb{B}_n$. The next result characterizes those functions in $H^\infty(\mathbb{B}_n)$ which are the image of elements in the unit ball of F^∞.

THEOREM 2.8: *Let F be a complex-valued function defined on $\mathbb{B}_n$, such that $|F(\zeta)| < 1$ for all $|\zeta| < 1$. Then there is $f \in F^\infty$, $\|f\|_\infty \le 1$ such that $f(\zeta) = F(\zeta)$, $\zeta \in \mathbb{B}_n$, if and only if for each $k \ge 1$ and each k-tuple of points $\lambda_1, \ldots, \lambda_k \in \mathbb{B}_n$, the matrix*

$$\left[\frac{1 - F(\lambda_j)\overline{F(\lambda_i)}}{1 - \langle \lambda_j, \lambda_i \rangle}\right]_{i,j=1,\ldots,k} \tag{2.8}$$

is positive definite. In particular, if (2.8) holds, then F is analytic on $\mathbb{B}_n$.

Proof: The necessity of (2.8) follows immediately from Theorem 2.4. Conversely, suppose that F satisfies (2.8). Let $\{\lambda_j\}_{j=1}^\infty$ be a countable dense set in $\mathbb{B}_n$. According to Theorem 2.4, for each k, there is $f_k \in F^\infty$ with $\|f_k\|_\infty \leq 1$ and

$$f_k(\lambda_j) = F(\lambda_j) \quad \text{for any } j = 1, \ldots, k. \tag{2.9}$$

Since $\{f_k\}_{k=1}^\infty$ is bounded and F^∞ is a dual space, according to Alaoglu's theorem, there is a subsequence $\{f_{k_m}\}_{k=1}^\infty$ such that f_{k_m} converges in the w^*-topology to an element $f \in F^\infty$, $\|f\|_\infty \leq 1$. Since w^* and WOT topologies coincide on F^∞ and the F^∞-functional calculus for C_0-contractions is WOT-continuous, we infer that

$$\lim_{m\to\infty} f_{k_m}(\lambda_{j1}, \ldots, \lambda_{jn}) = f(\lambda_{j1}, \ldots, \lambda_{jn}), \quad \text{where } \lambda_j = (\lambda_{j1}, \ldots, \lambda_{jn}).$$

Using (2.9), we have $\lim_{m\to\infty} f_{k_m}(\lambda_{j1}, \ldots, \lambda_{jn}) = F(\lambda_{j1}, \ldots, \lambda_{jn})$. Therefore, $f(\lambda_j) = F(\lambda_j)$ for any $j = 1, 2, \ldots$.

We claim that $f(\zeta) = F(\zeta)$ for any $\zeta \in \mathbb{B}_n$. Let λ be an arbitrary point in $\mathbb{B}_n$. By repeating the preceding argument, there is $g \in F^\infty$, $\|g\| \leq 1$ so that $g(\zeta) = F(\zeta)$ on the set $\{\lambda_j\}_{j=1}^\infty \cup \{\lambda\}$. Since the maps $\zeta \mapsto g(\zeta)$ and $\zeta \mapsto f(\zeta)$ are analytic in $\mathbb{B}_n$ and coincide on $\{\lambda_j\}_{j=1}^\infty$, which is dense in $\mathbb{B}_n$, we infer that they coincide on $\mathbb{B}_n$. In particular, we obtain $f(\lambda) = F(\lambda)$. Since λ was an arbitrary point in in $\mathbb{B}_n$, we deduce that f and F coincide on $\mathbb{B}_n$. In particular, $\zeta \mapsto F(\zeta)$ is a bounded analytic function in $\mathbb{B}_n$. This completes the proof. ∎

Condition (2.7) is necessary and sufficient for interpolation in F^∞ but only sufficient for interpolation in $H^\infty(\mathbb{B}_n)$. One can use the classical Cauchy formula for $\mathbb{B}_n$ to obtain a necessary condition for Nevanlinna–Pick interpolation in $H^\infty(\mathbb{B}_n)$. Recall that for every $f \in H^\infty(\mathbb{B}_n)$ and $\lambda \in \mathbb{B}_n$,

$$f(\lambda) = \int_{\partial\mathbb{B}_n} \frac{f(w)}{(1 - \langle \lambda, w \rangle)^n} d\sigma(w)$$

where σ is the rotation invariant probability measure on $\partial\mathbb{B}_n$. Using this formula, and a standard argument (e.g. like the one used in Section 3 of [CW]) we can check that if there exists $f \in H^\infty(\mathbb{B}_n)$, $\|f\|_\infty \leq 1$, such that $f(\lambda_j) = w_j$ for $j = 1, \ldots, k$, then

$$\left[\frac{1 - w_j \bar{w}_i}{(1 - \langle \lambda_j, \lambda_i \rangle)^n}\right]_{i,j=1,\ldots,k} \tag{2.10}$$

is positive definite. Now, one can easily check that the scalar version of condition (2.7) is stronger than condition (2.10) (see for example Lemma 4.1 of [CW]).

Let J be a w^*-closed, 2-sided ideal of F^∞. For any cardinal γ, the algebra $M_\gamma(F^\infty)$ is w^*-closed in $B(\oplus_\gamma \mathcal{F}^2)$ and $M_\gamma(J)$ is a w^*-closed, 2-sided ideal of $M_\gamma(F^\infty)$. Recently, Bercovici [B] proved that if the commutant of a w^*-closed subspace of $B(\mathcal{H})$ contains two isometries with orthogonal ranges, then the subspace has property $X_{0,1}$, which is stronger than property $\mathbb{A}_{\aleph_0}(1)$. One can use this result to show that $M_\gamma(F^\infty)$ has property $\mathbb{A}_1(1)$.

Another consequence of Theorem 2.1 is the following.

THEOREM 2.9: *For any cardinal γ, the map Φ: $M_\gamma(F^\infty)/M_\gamma(J) \to M_\gamma(B(\mathcal{N}_J))$ defined by*

$$\Phi([f_{\alpha\beta}] + M_\gamma(J)) = [P_{\mathcal{N}_J} f_{\alpha\beta}|_{\mathcal{N}_J}]$$

is an isometry.

Proof: It is enough to show that

$$\operatorname{dist}([f_{\alpha\beta}], M_\gamma(J)) = \|[P_{U\mathcal{N}_J} U^* f_{ij}(S_1, \ldots, S_n) U|_{U\mathcal{N}_J}]\|,$$

where U: $\mathcal{F}^2 \to \mathcal{F}^2$ is the flipping operator. For each $[g_{\alpha\beta}] \in M_\gamma(J)$, we have

$$\|[f_{\alpha\beta} + g_{\alpha\beta}]\| = \|[U^*(f_{\alpha\beta} + g_{\alpha\beta})U]\| \geq \|[P_{U\mathcal{N}_J} U^*(f_{\alpha\beta} + g_{\alpha\beta})U|_{U\mathcal{N}_J}]\|.$$

Since $g_{\alpha\beta} \in J$, according to Proposition 1.3, we have $P_{\mathcal{N}_J} g_{\alpha\beta}|_{\mathcal{N}_J} = 0$. Since U is an unitary operator with $U = U^*$, it is easy to see that $P_{U\mathcal{N}_J} U^* g_{\alpha\beta} U|_{U\mathcal{N}_J} = 0$. Combining this with the above inequality, we obtain

$$\operatorname{dist}([f_{\alpha\beta}], M_\gamma(J)) \geq \|[P_{U\mathcal{N}_J} U^* f_{\alpha\beta} U|_{U\mathcal{N}_J}]\|. \tag{2.11}$$

It remains to prove the converse inequality. Since $U^* f_{\alpha\beta} U$ commutes with $S_1, \ldots, S_n$, and $U\mathcal{N}_J$ is invariant to $S_1^*, \ldots, S_n^*, (U^* f_{\alpha\beta} U)^*$, it is clear that

$$[P_{U\mathcal{N}_J} U^* f_{\alpha\beta} U|_{U\mathcal{N}_J}]$$

commutes with $P_{U\mathcal{N}_J} S_i|_{U\mathcal{N}_J} \otimes I_\gamma$ for each $i = 1, 2, \ldots, n$. We can apply Theorem 2.1 to find $[\psi_{\alpha\beta}] \in M_\gamma(F^\infty)$ such that $\|[\psi_{\alpha\beta}]\| = \|[P_{U\mathcal{N}_J} U^* f_{\alpha\beta} U|_{U\mathcal{N}_J}]\|$ and $[P_{U\mathcal{N}_J} U^* \psi_{\alpha\beta} U|_{U\mathcal{N}_J}] = [P_{U\mathcal{N}_J} U^* f_{\alpha\beta} U|_{U\mathcal{N}_J}]$. According to Proposition 1.3, we infer that $[\phi_{\alpha\beta}] := [\psi_{\alpha\beta} - f_{\alpha\beta}] \in M_\gamma(J)$. Therefore,

$$\|[P_{U\mathcal{N}_J} U^* f_{\alpha\beta} U|_{U\mathcal{N}_J}]\| = \|[f_{\alpha\beta} + \phi_{\alpha\beta}]\| \geq \operatorname{dist}([f_{\alpha\beta}], M_\gamma(J)).$$

Combining this with (2.11), we complete the proof. ∎

Recall that for each $k \geq 1$, $M_k(F^\infty/J) = M_k(F^\infty)/M_k(J)$ (see [R]). Hence, as an immediate consequence of Theorem 2.9, we obtain the following.

COROLLARY 2.10: *The map* $\Phi: F^\infty/J \to P_{\mathcal{N}_J} F^\infty|_{\mathcal{N}_J}$ *given by* $\Phi(f) = P_{\mathcal{N}_J} f|_{\mathcal{N}_J}$ *is a completely isometric representation.*

Let $\mathcal{P}_m$ be the set of all polynomials in $\mathcal{F}^2$ of degree $\leq m$, and denote $\mathcal{F}^\infty_{>m} = \mathcal{F}^\infty \cap (\mathcal{F}^2 \ominus \mathcal{P}_m)$. Setting $J = \mathcal{F}^\infty_{>m}$ in Corollary 2.10, we can deduce the Carathéodory interpolation theorem on Fock spaces [Po7].

COROLLARY 2.11: *Let* $p \in \mathcal{P}_m$ *be fixed. Then*

$$\operatorname{dist}(p, \mathcal{F}^\infty_{>m}) = \|P_{\mathcal{P}_m} p(S_1, \ldots, S_n)|_{\mathcal{P}_m}\|.$$

Let us remark that Theorem 2.9 is no longer true if we replace F^∞ by the noncommutative disk algebra $\mathcal{A}_n$ and J is a closed, 2-sided ideal of $\mathcal{A}_n$. To see this, let $\lambda \in \mathbb{C}^n$ be of norm one and let $J := \{\psi \in \mathcal{A}_n: \psi(\lambda_1, \ldots, \lambda_n) = 0 \text{ and } \langle \psi, 1\rangle = 0\}$. It is easy to see that $\mathcal{N}_J$ is the span of 1 and so is $\mathcal{N}_{J_w}$ (see also Example 3.6). Then $J_w = \{\psi \in F^\infty: \langle\psi, 1\rangle = 0\}$. If one takes a polynomial $p \in \mathcal{P}$ such that $\langle p, 1\rangle = 0$ but $p(\lambda_1, ..., \lambda_n) \neq 0$, then $\operatorname{dist}(p, J) > 0$ but $\operatorname{dist}(p, J_w) = 0$. Therefore,

$$\operatorname{dist}(p, J) \neq \operatorname{dist}(p, J_w) = \|P_{\mathcal{N}_{J_w}} f|_{\mathcal{N}_{J_w}}\| = \|P_{\mathcal{N}_J} f|_{\mathcal{N}_J}\|.$$

However, we will show that $\mathcal{A}_n/J$ is completely isometrically isomorphic to $P_{\mathcal{N}_J}\mathcal{A}_n|_{\mathcal{N}_J}$, for certain closed ideals J of $\mathcal{A}_n$.

PROPOSITION 2.12: *Let* $\lambda_1, \ldots, \lambda_k \in \mathbb{B}_n$ *and define*

$$J = \{\varphi \in \mathcal{A}_n: \varphi(\lambda_j) = 0 \text{ for every } j = 1, 2, \ldots, k\}, \text{ and}$$
$$J_w = \{\varphi \in F^\infty: \varphi(\lambda_j) = 0 \text{ for every } j = 1, 2, \ldots, k\}.$$

Then the map $\Psi: \mathcal{A}_n/J \to P_{\mathcal{N}_J}\mathcal{A}_n|_{\mathcal{N}_J}$ *defined by* $\Psi(f + J) = P_{\mathcal{N}_J} f|_{\mathcal{N}_J}$ *is a completely isometric representation.*

Proof: According to Corollary 2.10 and Lemma 1.4, for any $f \in \mathcal{A}_n$,

$$\operatorname{dist}(f, J_w) = \| P_{\mathcal{N}_{J_w}} f|_{\mathcal{N}_{J_w}} \| = \|\mathcal{P}_{\mathcal{N}_J} f|_{\mathcal{N}_J}\|.$$

Therefore, it is enough to prove that $\operatorname{dist}(f, J_w) = \operatorname{dist}(f, J)$. Let us define $\Phi: \mathcal{A}_n/J \to F^\infty/J_w$ by $\Phi(\varphi + J) = \varphi + J_w$. Notice that Φ is contractive. We shall prove that for every $\varphi \in F^\infty$ with $\|\varphi + J_w\| = 1$, there exists $\psi \in \mathcal{A}_n$ such that $\|\psi + J\| = 1$ and $\Phi(\psi + J) = \varphi + J_w$.

Assume that $\|\varphi\| = \|\varphi + J_w\| = 1$ and find $\varphi_k \in \mathcal{A}_n$ such that $\|\varphi_k\| \leq 1$ and $\varphi_k \to \varphi$ in the WOT. Since $\mathcal{A}_n/J$ is finite dimensional, we assume (after passing to a subsequence) that $\varphi_k + J$ converges to $\psi + J$ in the norm of $\mathcal{A}_n/J$ for some $\psi \in \mathcal{A}_n$. Then $\|\psi + J\| \leq 1$ and there exists a sequence $\eta_k \in J$ such that $\varphi_k + \eta_k \to \psi$ in the norm topology of $\mathcal{A}_n$. Then $\eta_k = (\eta_k + \varphi_k) - \varphi_k \to \psi - \varphi$ in the WOT of F^∞. Since $\eta_k \in J \subset J_w$ for each k and since J_w is WOT-closed, we have that $\psi - \varphi \in J_w$. Therefore, $\varphi + J_w = \psi + J_w = \Phi(\psi + J)$. Since $\dim \mathcal{A}_n/J = \dim F^\infty/J_w$, it is clear now that Φ is isometric. The argument works also when passing to matrices, so the map Φ is completely isometric. ∎

Combining Theorem 2.4 with Proposition 2.12, we infer the following Nevanlinna–Pick interpolation theorem for the noncommutative disc algebra $\mathcal{A}_n$. For simplicity, we consider only the scalar case.

Corollary 2.13: *Let $\lambda_1, \ldots, \lambda_k \in \mathbb{B}_n$, and $w_1, \ldots, w_k \in \mathbb{C}$. Then the matrix*

$$\left[\frac{1 - w_j \bar{w}_i}{1 - \langle \lambda_j, \lambda_i \rangle}\right]_{i,j=1,\ldots,k}$$

is positive definite if and only if for any $\epsilon > 0$ there exists $f \in \mathcal{A}_n$, $\|f\|_\infty \leq 1 + \epsilon$, such that $f(\lambda_j) = w_j$ for every $j = 1, \ldots, k$.

3. Poisson transforms and von Neumann inequalities

In [Po9], the second author found an elementary proof of the inequality (1) based on noncommutative Poisson transforms associated to row contractions. In this section, we will recall this construction (see [Po9, Section 8]) in a particular case and use it to obtain new results.

As in [Po1], $T = [T_1, \ldots, T_n]$ is called C_0-contraction if T is a contraction and

$$\text{SOT-}\lim_{k\to\infty} \sum_{\alpha \in \mathbb{F}_n^+, |\alpha| = k} T_\alpha T_\alpha^* = 0. \tag{3.1}$$

Recall that the sequence $\{\sum_{|\alpha|=k} T_\alpha T_\alpha^* \colon k \geq 0\}$ of positive operators is non-increasing, and that (3.1) holds if and only if $\sum_{|\alpha|=k} \|T_\alpha^* h\|^2 \to 0$ for every $h \in \mathcal{H}$.

Suppose that $T = [T_1, \ldots, T_n]$ is a C_0-contraction and let

$$\Delta := (I_{\mathcal{H}} - \sum_{i=1}^n T_i T_i^*)^{\frac{1}{2}}.$$

Since

$$\sum_{|\alpha|=k} T_\alpha \Delta^2 T_\alpha^* = \sum_{|\alpha|=k} T_\alpha T_\alpha^* - \sum_{|\alpha|=k+1} T_\alpha T_\alpha^*,$$

it is clear that $\sum_{\alpha\in\mathbb{F}_n^+} T_\alpha\Delta^2T_\alpha^* = I_\mathcal{H} - \lim_{k\to\infty}\sum_{|\alpha|=k+1} T_\alpha T_\alpha^* = I_\mathcal{H}$.

The Poisson kernel $K = K(T)$ associated to $T = [T_1,\dots,T_n]$ is the linear map

$$K\colon \mathcal{H} \to \mathcal{F}^2\otimes\mathcal{H} \qquad \text{defined by} \qquad Kh = \sum_{\alpha\in\mathbb{F}_n^+} e_\alpha\otimes\Delta T_\alpha^* h.$$

Since $\sum_\alpha T_\alpha\Delta^2T_\alpha^* = I_\mathcal{H}$, K is an isometry. It is easy to check that, for each $\alpha\in\mathbb{F}_n^+$, $(S_\alpha^*\otimes I)Kh = KT_\alpha^*h$. Hence, for every $\alpha,\beta\in\mathbb{F}_n^+$,

$$K^*\big[S_\alpha S_\beta^*\otimes I\big]K = T_\alpha T_\beta^*. \tag{3.2}$$

The map Ψ: $B(\mathcal{F}^2)\to B(\mathcal{H})$ defined by $\Psi(A) = K^*[A\otimes I]K$ is clearly unital, completely contractive (hence, completely positive), and w^*-continuous. Moreover, for each $\alpha,\beta\in\mathbb{F}_n^+$, $\Psi(S_\alpha S_\beta^*) = T_\alpha T_\beta^*$. The restriction of Ψ to F^∞, which is denoted by Ψ_T, provides a WOT-continuous F^∞-functional calculus for the C_0-contractions $T = [T_1,\dots,T_n]$, which is a particular case of [Po6]. That is,

$$\Psi_T\colon F^\infty\to B(\mathcal{H}) \quad \text{satisfies} \quad \Psi_T(\varphi(S_1,\dots,S_n)) = \varphi(T_1,\dots,T_n) \tag{3.3}$$

for every $\varphi\in F^\infty$.

Suppose now that $T = [T_1,\dots,T_n]$ is a row contraction. For each $0<r<1$, let $K_r = K_r(T)$ be the Poisson kernel associated to $[rT_1,\dots rT_n]$, which is clearly a C_0-contraction. Let $C^*(S_1,\dots,S_n)$ be the C^*-algebra generated by $S_1,\dots,S_n$, the extension through compacts of the Cuntz algebra $\mathcal{O}_n$ (see [Cu]). The Poisson transform associated to $T = [T_1,\dots,T_n]$ is the linear map

$$\Phi_T\colon C^*(S_1,\dots,S_n)\to B(\mathcal{H}) \quad \text{defined by} \quad \Phi_T(f) = \lim_{r\to 1} K_r^*[f\otimes I]K_r \tag{3.4}$$

(in the uniform topology of $B(\mathcal{H})$). It is easy to see that Φ_T is unital, completely contractive, and for every $\alpha,\beta\in\mathbb{F}_n^+$, $\Phi_T(S_\alpha S_\beta^*) = T_\alpha T_\beta^*$. Inequality (1) from the introduction follows by restricting Φ_T to $\mathcal{A}_n$.

A simple consequence of the noncommutative Poisson transform is the following result which turns out to be crucial for the rest of this paper.

PROPOSITION 3.1: *Let $T = [T_1,\dots,T_n]$ be a C_0-contraction with its Poisson kernel K, and let $\mathcal{N}$ be a subspace of $\mathcal{F}^2$ invariant under $S_1^*,\dots,S_n^*$. If K takes values in $\mathcal{N}\otimes\mathcal{H}$, then there exists a unital, completely contractive, w^*-continuous map Φ: $B(\mathcal{N})\to B(\mathcal{H})$ such that for every $\alpha,\beta\in\mathbb{F}_n^+$, $\Phi(B_\alpha B_\beta^*) = T_\alpha T_\beta^*$, where $B_k = P_\mathcal{N}S_k|_\mathcal{N}$ for any $k = 1,\dots,n$.*

Proof: Since $\mathcal{N}\subset\mathcal{F}^2$ is an invariant subspace of $S_1^*,\dots,S_n^*$, for every $\alpha,\beta\in\mathbb{F}_n^+$, $P_\mathcal{N}S_\alpha S_\beta^*|_\mathcal{N} = B_\alpha B_\beta^*$. By hypothesis, $K = (P_\mathcal{N}\otimes I)K$. Hence, and according to

(3.2), for each $\alpha,\beta\in\mathbb{F}_n^+$, we have

$$\begin{aligned}T_\alpha T_\beta^* &= K^*[S_\alpha S_\beta^*\otimes I]K = K^*(P_{\mathcal{N}}\otimes I)[S_\alpha S_\beta^*\otimes I](P_{\mathcal{N}}\otimes I)K\\ &= K^*[P_{\mathcal{N}}S_\alpha S_\beta^* P_{\mathcal{N}}\otimes I]K = K^*[B_\alpha B_\beta^*\otimes I]K.\end{aligned}\tag{3.5}$$

To complete the proof, define $\Phi\colon B(\mathcal{N})\to B(\mathcal{H})$ by $\Phi(A)=K^*[A\otimes I]K$. ∎

Remark 3.2: If $T=[T_1,\dots,T_n]$ is a contraction and its Poisson kernel K_r takes values in $\mathcal{N}\otimes\mathcal{H}$ for every $0<r<1$, then there is a unital, completely contractive map $\Phi\colon C^*(B_1,\dots,B_n)\to B(\mathcal{H})$ satisfying $\Phi(B_\alpha B_\beta^*)=T_\alpha T_\beta^*$ for all $\alpha,\beta\in\mathbb{F}_n^+$.

Proof: It follows from (3.5) that

$$\lim_{r\to1}K_r^*[B_\alpha B_\beta^*\otimes I]K_r = \lim_{r\to1} r^{|\alpha|}T_\alpha r^{|\beta|}T_\beta^* = T_\alpha T_\beta^*.$$

Hence, the map $B_\alpha B_\beta^*\mapsto T_\alpha T_\beta^*$, defined on $\operatorname{span}\{B_\alpha B_\beta^* : \alpha,\beta\in\mathbb{F}_n^+\}$, is completely contractive. By [Ar1], it can be extended to a unital, completely contractive map $\Phi\colon C^*(B_1,\dots,B_n)\to B(\mathcal{H})$ satisfying $\Phi(B_\alpha B_\beta^*)=T_\alpha T_\beta^*$ for all $\alpha,\beta\in\mathbb{F}_n^+$. ∎

To illustrate Proposition 3.1 and Remark 3.2, we will consider a row contraction $T=[T_1,\dots,T_n]$ satisfying the following commutation relations:

$$T_jT_i=\lambda_{ji}T_iT_j \quad \text{for every } 1\le i<j\le n,\tag{3.6}$$

where $\lambda_{ij}\in\mathbb{C}$ for $1\le i<j\le n$.

Example 3.3: There exists a subspace $\mathcal{N}=\mathcal{N}(\{\lambda_{ij}\})$ of $\mathcal{F}^2$, invariant under $S_1^*,\dots,S_n^*$, such that the operators $B_k=P_{\mathcal{N}}S_k|_{\mathcal{N}}$, $k=1,\dots,n$, satisfy (3.6) and for every row contraction $T=[T_1,\dots,T_n]$ satisfying (3.6), there exists a unital completely contractive linear map $\Phi\colon C^*(B_1,\dots,B_n)\to B(\mathcal{H})$ such that $\Phi(B_\alpha B_\beta^*)=T_\alpha T_\beta^*$ for any $\alpha,\beta\in\mathbb{F}_n^+$.

Proof: Fix $k\in\mathbb{N}$, and consider $\alpha=g_{i_1}g_{i_2}\cdots g_{i_k}\in\mathbb{F}_n^+$ satisfying $i_1\le i_2\le\cdots\le i_k$, and a permutation $\pi\in\Pi_k$ on $\{1,2,\dots,k\}$. Then, from (3.6),

$$T_{\pi(\alpha)}=\epsilon_{\pi(\alpha)}T_\alpha, \quad \text{where } \epsilon_{\pi(\alpha)}:=\prod_{\substack{j<\ell\\ \pi(j)>\pi(\ell)}}\lambda_{i_{\pi(j)}i_{\pi(\ell)}}$$

and $\pi(\alpha):=g_{i_{\pi(1)}}g_{i_{\pi(2)}}\cdots g_{i_{\pi(k)}}$. Let $\mathcal{N}(\{\lambda_{ij}\})$ be the subspace of $\mathcal{F}^2$ defined by

$$\mathcal{N}(\{\lambda_{ij}\}):=\overline{\operatorname{span}}\left\{\sum_{\pi\in\Pi_k}\bar{\epsilon}_{\pi(\alpha)}e_{\pi(\alpha)}\colon \alpha=g_{i_1}g_{i_2}\cdots g_{i_k}\in\mathbb{F}_n^+, i_1\le\cdots\le i_k, k\in\mathbb{N}\right\}.$$

It is easy to see that if $T = [T_1, \ldots, T_n]$ is a C_0-contraction and satisfies (3.6), then its Poisson kernel takes values in $\mathcal{N}(\{\lambda_{ij}\}) \otimes \mathcal{H}$. Indeed,

$$Kh = \sum_{k=0}^{\infty} \sum_{\substack{\alpha=g_{i_1}\cdots g_{i_k} \\ i_1 \le \cdots \le i_k}} \sum_{\pi \in \Pi_k} e_{\pi(\alpha)} \otimes \Delta T^*_{\pi(\alpha)} h = \sum_{k=0}^{\infty} \sum_{\substack{\alpha=g_{i_1}\cdots g_{i_k} \\ i_1 \le \cdots \le i_k}} v_\alpha \otimes \Delta T^*_\alpha h,$$

where $v_\alpha = \sum_{\pi \in \Pi_k} \bar{\epsilon}_{\pi(\alpha)} e_{\pi(\alpha)} \in \mathcal{N}(\{\lambda_{ij}\})$. One can verify directly, from the definition of $\mathcal{N}(\{\lambda_{ij}\})$, that this space is invariant under $S_1^*, \ldots, S_n^*$, although it is easier to check that $\mathcal{N}(\{\lambda_{ij}\}) = \mathcal{N}_J$, where J is the WOT-closed, 2-sided ideal in $\mathcal{F}^\infty$ generated by $\{e_j \otimes e_i - \lambda_{ji} e_i \otimes e_j \colon 1 \le i < j \le n\}$. Then, from Proposition 1.3, the B_k's satisfy (3.6). The rest of the statement of Example 3.3 is an immediate consequence of Remark 3.2. ∎

The case where $\lambda_{ji} = 1$ for $1 \le i < j \le n$ appears in [Ath], [Po9], and [Ar2]. In this situation, condition (3.6) means that the T_i's are commuting and $\mathcal{N}(\{\lambda_{ji}\})$ is the symmetric Fock space. If $\lambda_{ji} = -1$ for $1 \le i < j \le n$, then the T_i's are anti-commuting and $\mathcal{N}(\{\lambda_{ji}\})$ is the anti-symmetric Fock space.

Example 3.4: If J_k is a WOT-closed, 2-sided ideal generated by some elements in $\operatorname{span}\{e_\alpha \colon |\alpha| = k\}$, then a similar result to Example 3.3 holds for any contraction $T = [T_1, \ldots, T_n]$ such that $\phi(T_1, \ldots, T_n) = 0$ for each $\phi \in J_k$.

In Section 4, we will consider the F^∞-functional calculus associated to row contractions satisfying (3.6), or as in Example 3.4.

Let $\phi \in F^\infty$ and let J_ϕ be the WOT-closed, 2-sided ideal generated by ϕ in F^∞. If $\mathcal{N}_{J_\phi} \ne \{0\}$, then there is a nontrivial C_0-contraction $T = [T_1, \ldots, T_n]$ such that $\phi(T_1, \ldots, T_n) = 0$. Indeed, define $T_i := P_{\mathcal{N}_{J_\phi}} S_i|_{\mathcal{N}_{J_\phi}}$. According to [Po1], it is clear that $T = [T_1, \ldots, T_n]$ is a C_0-contraction, since the F^∞-functional calculus associated to C_0-contractions is WOT-continuous. It is easy to see that $\phi(T_1, \ldots, T_n) = P_{\mathcal{N}_{J_\phi}} \phi(S_1, \ldots, S_n)|_{\mathcal{N}_{J_\phi}} = 0$ (see also Lemma 4.4).

LEMMA 3.5: *Suppose that $T = [T_1, \ldots, T_n]$ is a C_0-contraction with its Poisson kernel K, and that J is a* WOT*-closed, 2-sided ideal of F^∞ such that for every $\varphi \in J$, $\varphi(T_1, \ldots, T_n) = 0$. Then K takes values in $\mathcal{N}_J \otimes \mathcal{H}$. Consequently, $\mathcal{N}_J \ne (0)$.*

Proof: For any polynomial $p \in \mathcal{P}$, $p = \sum_\alpha a_\alpha e_\alpha$, we have

$$\begin{aligned} \langle Kk, p \otimes h \rangle &= \sum_\alpha \bar{a}_\alpha \langle k, T_\alpha \Delta h \rangle = \left\langle k, \left(\sum_\alpha a_\alpha T_\alpha \right) \Delta h \right\rangle \\ &= \langle k, p(T_1, \ldots, T_n) \Delta h \rangle \end{aligned}$$

for any $h, k \in \mathcal{H}$. Since the F^∞-functional calculus for C_0-contractions is WOT-continuous and $\mathcal{P}$ is WOT-dense in F^∞ we deduce that for any $\varphi \in J$ and $h, k \in \mathcal{H}$,

$$\langle Kk, \varphi \otimes h \rangle = \langle k, \varphi(T_1, \ldots, T_n)\Delta h \rangle = 0.$$

Since $\mathcal{M}_J$ is the closure of J in $\mathcal{F}^2$, we see that for every $k \in \mathcal{H}$,

$$Kk \in \left(\mathcal{M}_J \otimes H\right)^\perp = \mathcal{N}_J \otimes H.$$

This completes the proof. ∎

If $T = [T_1, \ldots, T_n]$ is a C_0-contraction, then

$$J_1 := \{\varphi \in F^\infty \colon \varphi(T_1, \ldots, T_n) = 0\} = \operatorname{Ker} \Psi_T$$

is a WOT-closed, 2-sided ideal of F^∞. Similarly,

$$J_2 := \{\varphi \in \mathcal{A}_n \colon \varphi(T_1, \ldots, T_n) = 0\} = \operatorname{Ker} \Phi_T$$

is a closed, 2-sided ideal of $\mathcal{A}_n$. Lemma 3.5 is stated for F^∞, but it holds true also for $\mathcal{A}_n$. Therefore $\mathcal{N}_{J_1} \neq \{0\}$ and $\mathcal{N}_{J_2} \neq \{0\}$. Let us remark that if $[T_1, \ldots, T_n]$ is just a contraction (not necessarily C_0), then $\mathcal{N}_{J_2}$ may be zero.

Example 3.6: (Point evaluations) Let $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, n$, be such that $\sum_{i=1}^n |\lambda_i|^2 < 1$. Then $\lambda = [\lambda_1, \ldots, \lambda_n]$ is a C_0-contraction, and hence, $J_w := \{\varphi \in F^\infty : \varphi(\lambda_1, \ldots, \lambda_n) = 0\}$ is a WOT-closed, 2-sided ideal of F^∞. It is known that $\mathcal{N}_{J_\lambda} = \operatorname{span}\{z_\lambda\}$ where $z_\lambda = 1 + \sum_{k \geq 1} (\lambda_1 e_1 + \cdots + \lambda_n e_n)^{\otimes k}$ and $\varphi(\lambda_1, \ldots, \lambda_n) = \langle \varphi, z_\lambda \rangle$ for every $\varphi \in F^\infty$ (see [APo], [Ar2], and [DP1]). Notice that if $\sum_{i=1}^n |\lambda_i|^2 = 1$, then $J = \{\varphi \in \mathcal{A}_n : \varphi(\lambda_1, \ldots, \lambda_n) = 0\}$ is a closed, 2-sided ideal of $\mathcal{A}_n$ but one can check that $\mathcal{N}_J = \{0\}$.

Combining Proposition 3.1 and Lemma 3.5, we obtain the following.

THEOREM 3.7: *Let $T = [T_1, \ldots, T_n]$ be a C_0-contraction, and let J be a WOT-closed, 2-sided ideal of F^∞ such that for every $\varphi \in J$, $\varphi(T_1, \ldots, T_n) = 0$, then there exists a unital, completely contractive, w^*-continuous map Φ: $B(\mathcal{N}_J) \to B(\mathcal{H})$ such that for every $\alpha, \beta \in \mathbb{F}_n^+$, $\Phi(B_\alpha B_\beta^*) = T_\alpha T_\beta^*$, where $B_k = P_{\mathcal{N}_J} S_k|_{\mathcal{N}_J}$, $k = 1, \ldots, n$.*

One can easily see that there is an $\mathcal{A}_n$-version of this theorem corresponding to closed, 2-sided ideals in $\mathcal{A}_n$, $J \subset \operatorname{Ker} \Phi_T$ with $\mathcal{N}_J \neq \{0\}$. Let $T = [T_1, \ldots, T_n]$ be a contraction, and let $J \subset \operatorname{Ker} \Phi_T$ be a closed, 2-sided ideal of $\mathcal{A}_n$ such that $\mathcal{N}_J \neq \{0\}$. Notice that Remark 3.2 holds true if we take $\mathcal{N} = \mathcal{N}_J$.

Given $T = [T_1, \ldots, T_n]$ a C_0-contraction with Poisson kernel K, the best von Neumann inequality given by Proposition 3.1 comes from the smallest subspace $\mathcal{N}_T$ of $\mathcal{F}^2$ which is invariant under $S_1^*, \ldots, S_n^*$ and such that K takes values in $\mathcal{N}_T \otimes \mathcal{H}$. It is not hard to see that $\mathcal{N}_T = \overline{\text{span}}\,\{\langle T_\alpha^* k, \Delta h\rangle e_\alpha : h, k \in \mathcal{H}; \alpha \in \mathbb{F}_n^+\}$. First notice that $\mathcal{N}_T$ is the smallest $\mathcal{N}$ such that K takes values in $\mathcal{N} \otimes \mathcal{F}^2$, and then notice that $\mathcal{N}_T$ is invariant under $S_1^*, \ldots, S_n^*$.

4. $\mathcal{W}(B_1, ..., B_n)$ and F^∞/J-functional calculus for row contractions

In this section J will be a w^*-closed, 2-sided ideal of F^∞. Recall that $\mathcal{N}_J$ is the orthogonal complement of the image of J in $\mathcal{F}^2$ and that $B_k = P_{\mathcal{N}_J} S_k|_{\mathcal{N}_J}$ for $k = 1, \ldots, n$. We define $\mathcal{W}(B_1, \ldots, B_n)$ to be w^*-closure of of the algebra generated by the B_k's and the identity.

We will prove that F^∞/J is canonically isomorphic to $\mathcal{W}(B_1, \ldots, B_n)$. We will describe the commutant of $\mathcal{W}(B_1, \ldots, B_n)$ and will show that $\mathcal{W}(B_1, \ldots, B_n)$ is the double commutant of $\{B_1, \ldots, B_n\}$. We will show that $\mathcal{W}(B_1, \ldots, B_n)$ has the $\mathbb{A}_1(1)$ property and hence the w^* and WOT topologies agree on this algebra. Finally, we will develop a F^∞/J-functional calculus for row contractions.

A direct consequence of Proposition 1.2 and Corollary 2.10 is the following.

THEOREM 4.1: *The map $\Psi: F^\infty/J \to B(\mathcal{N}_J)$ defined by*

$$\Psi(\varphi + J) = P_{\mathcal{N}_J} \varphi(S_1, \ldots, S_n)|_{\mathcal{N}_J}$$

is a completely isometric isomorphism onto $P_{\mathcal{N}_J} F^\infty|_{\mathcal{N}_J}$, and a homeomorphism relative to the w^*-topology on F^∞/J and the WOT*-topology on $P_{\mathcal{N}_J} F^\infty|_{\mathcal{N}_J}$.*

Proof: Since the fact that Ψ is a completely isometric homomorphism was already proved in Corollary 2.10 (see also Section 5), we only have to prove that Ψ is a w^*-WOT homeomorphism.

By Proposition 1.2, $\varphi_i + J \to \varphi + J$ in the w^* topology iff for every $\xi_1, \xi_2 \in \mathcal{N}_J$, $\langle \varphi_i \otimes \xi_1, \xi_2\rangle \to \langle \varphi \otimes \xi_1, \xi_2\rangle$. This is clearly equivalent to $P_{\mathcal{N}_J} \varphi_i|_{\mathcal{N}_J} \to P_{\mathcal{N}_J} \varphi|_{\mathcal{N}_J}$ in the weak operator topology. ∎

Using again the noncommutative commutant lifting theorem [Po3], we can prove the following.

PROPOSITION 4.2: *The algebra $P_{\mathcal{N}_J} F^\infty|_{\mathcal{N}_J}$ is the* WOT*-closed algebra generated by $P_{\mathcal{N}_J} S_i|_{\mathcal{N}_J}$, $i = 1, \ldots, n$, and the identity. Moreover, we have*

$$P_{\mathcal{N}_J} F^\infty|_{\mathcal{N}_J} = \{P_{\mathcal{N}_J} U^* F^\infty U|_{\mathcal{N}_J}\}' = \{P_{\mathcal{N}_J} F^\infty|_{\mathcal{N}_J}\}''. \tag{4.1}$$

Proof: We first show that $P_{\mathcal{N}_J}F^\infty|_{\mathcal{N}_J}$ is weakly closed. Notice that $\mathcal{N}_J$ is an invariant subspace of $U^*S_i^*U$ for each $i = 1, \ldots, n$, and

$$[P_{\mathcal{N}_J}U^*S_1U|_{\mathcal{N}_J}, \ldots, P_{\mathcal{N}_J}U^*S_nU|_{\mathcal{N}_J}]$$

is a C_0-contraction with the minimal isometric dilation $[U^*S_1U, \ldots, U^*S_nU]$. According to the commutant lifting theorem, $X \in \{P_{\mathcal{N}_J}U^*S_iU|_{\mathcal{N}_J} : i = 1, \ldots, n\}'$ if and only if $X = P_{\mathcal{N}_J}Y|_{\mathcal{N}_J}$ for some $Y \in \{U^*S_1U, \ldots, U^*S_nU\}'$. Using [Po7], we get $Y = f(S_1, \ldots, S_n)$, $f \in \mathcal{F}^\infty$. Now, it is clear that $P_{\mathcal{N}_J}F^\infty|_{\mathcal{N}_J} = \{P_{\mathcal{N}_J}U^*F^\infty U|_{\mathcal{N}_J}\}'$ and, hence, $P_{\mathcal{N}_J}F^\infty|_{\mathcal{N}_J}$ is a WOT-closed algebra.

Since the polynomials in $S_1, \ldots, S_n$ are WOT-dense in F^∞, it is clear that $P_{\mathcal{N}_J}F^\infty|_{\mathcal{N}_J}$ is the WOT-closed algebra generated by $P_{\mathcal{N}_J}S_i|_{\mathcal{N}_J}$, $i = 1, \ldots, n$, and the identity. The second equality in (4.1) follows in a similar manner. ∎

PROPOSITION 4.3: *The algebra $\mathcal{W}(B_1, \ldots, B_n)$ has property $\mathbb{A}_1(1)$. Moreover, $\mathcal{W}(B_1, \ldots, B_n) = P_{\mathcal{N}_J}F^\infty|_{\mathcal{N}_J}$.*

Proof: Let $f \in \mathcal{W}(B_1, \ldots, B_n)_*$ satisfy $\|f\| = 1$. Since

$$\mathcal{W}(B_1, \ldots, B_n)_* \equiv c_1(\mathcal{N}_J)/{}^\perp\mathcal{W}(B_1, \ldots, B_n),$$

for each $\epsilon > 0$, find $g \in c_1(\mathcal{N}_J)$ satisfying $\|g\| \le 1+\epsilon$ and $f = g + {}^\perp\mathcal{W}(B_1, \ldots, B_n)$. Let $i_{\mathcal{N}_J}\colon \mathcal{N}_J \to \mathcal{F}^2$ be the inclusion and notice that $(i_{\mathcal{N}_J})^* = P_{\mathcal{N}_J}$. It is easy to check that $i_{\mathcal{N}_J} \circ g \circ P_{\mathcal{N}_J} \in {}^\perp J \subset c_1(\mathcal{F}^2)$. Then, by Proposition 1.2, there exists $\varphi, \psi \in \mathcal{N}_J$ satisfying $\|\varphi\|_2\|\psi\|_2 \le (1+\epsilon)\|g\| \le (1+\epsilon)^2$ such that, for every $\eta \in F^\infty$, $\langle i_{\mathcal{N}_J} \circ g \circ P_{\mathcal{N}_J}, \eta\rangle = \langle \eta \otimes \varphi_1, \varphi_2\rangle$. Now, for each noncommutative polynomial $p \in \mathcal{P}$, we have

$$\begin{aligned}\langle f, p(B_1, \ldots, B_n)\rangle &= \langle g, p(B_1, \ldots, B_n)\rangle = \langle g, P_{\mathcal{N}_J}p(S_1, \ldots, S_n)|_{\mathcal{N}_J}\rangle \\ &= \langle i_{\mathcal{N}_J} \circ g \circ P_{\mathcal{N}_J}, p(S_1, \ldots, S_n)\rangle = \langle p \otimes \varphi, \psi\rangle \\ &= \langle p(S_1, \ldots, S_n)\varphi, P_{\mathcal{N}_J}\psi\rangle = \langle P_{\mathcal{N}_J}p(S_1, \ldots, S_n)|_{\mathcal{N}_J}\varphi, \psi\rangle \\ &= \langle p(B_1, \ldots, B_n)\varphi, \psi\rangle.\end{aligned}$$

Since f is w^*-continuous, we prove the $\mathbb{A}_1(1)$ property. The last part of the theorem follows from Proposition 4.2. ∎

Let J be the w^*-closed, 2-sided ideal of F^∞ generated by $S_2, S_3, \ldots, S_n$. It is easy to see that $\mathcal{N}_J$ is the closed span of $e_1^{\otimes k}$ for $k \ge 0$, and that $B_2 = \cdots = B_n = 0$. Hence, $\mathcal{W}(B_1, \ldots, B_n) = \mathcal{W}(B_1)$, where $B_1 e_1^{\otimes k} = e_1^{k+1}$. Since B_1 is a unilateral shift of multiplicity one, we use Proposition 4.3 to give an alternative proof of the well known fact that $\mathcal{W}(B_1)$ has property $\mathbb{A}_1(1)$. Moreover, it is also known (see [BFP, Theorem 4.16]) that $\mathcal{W}(B_1)$ does not even satisfy property $\mathbb{A}_2$. In that sense, Proposition 4.3 is best possible.

Let us recall from [Po1] that a contraction $[T_1, \dots, T_n]$ is called completely non-coisometric (c.n.c.) if there is no $h \in \mathcal{H}$, $h \neq 0$ such that

$$\sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = \|h\|^2, \quad \text{for any } k \in \{1, 2, \dots\}.$$

Let $T = [T_1, \dots, T_n]$ be a c.n.c. contraction and let

$$\Psi_T \colon F^\infty \to B(\mathcal{H}), \quad \Psi_T(f) = f(T_1, \dots, T_n),$$

be the F^∞-functional calculus associated to T. In this section, we prove that if J is a WOT-closed, 2-sided ideal of F^∞ with $J \subset \operatorname{Ker} \Psi_T$, then there is a WOT-continuous, F^∞/J-functional calculus associated to T.

LEMMA 4.4: *Let $B = [B_1, \dots, B_n]$ and let Ψ_B be the F^∞-functional calculus associated to it. Then*

$$\mathcal{W}(B_1, \dots, B_n) = \Psi_B(F^\infty) = \{f(B_1, \dots, B_n) : \ f \in F^\infty\}.$$

Proof: According to Proposition 4.3, it is enough to prove that

$$f(B_1, \dots, B_n) = P_{\mathcal{N}_J} f(S_1, \dots, S_n)|_{\mathcal{N}_J} \tag{4.2}$$

for any $f \in \mathcal{F}^\infty$. Since $B_i = P_{\mathcal{N}_J} S_i|_{\mathcal{N}_J}$, (4.2) holds for polynomials, and consequently for elements in the noncommutative disc algebra $\mathcal{A}_n$. Since $B = [B_1, \dots, B_n]$ is a C_0-contraction, according to the F^∞-functional calculus, we have

$$\begin{aligned} f(B_1, \dots, B_n) :&= \text{SOT-}\lim_{r\to 1} f_r(B_1, \dots, B_n) \\ &= \text{SOT-}\lim_{r\to 1} P_{\mathcal{N}_J} f_r(S_1, \dots, S_n)|_{\mathcal{N}_J} = P_{\mathcal{N}_J} f(S_1, \dots, S_n)|_{\mathcal{N}_J} \end{aligned}$$

for any $f \in F^\infty$. ∎

THEOREM 4.5: *Let $T = [T_1, \dots, T_n]$ be a c.n.c. contraction and let*

$$\Psi_T \colon F^\infty \to B(\mathcal{H}), \quad \Psi_T(f) = f(T_1, \dots, T_n),$$

be the F^∞-functional calculus associated to T. If J is a WOT-closed, 2-sided ideal of F^∞ with $J \subset \operatorname{Ker}\Psi_T$, then the map

$$\Psi_{T,J} \colon \mathcal{W}(B_1, \dots B_n) \to B(\mathcal{H}); \quad \Psi_{T,J}(f(B_1, \dots, B_n)) := f(T_1, \dots, T_n), \tag{4.3}$$

is a WOT*-continuous, completely contractive homomorphism.*

In particular, for any $f \in F^\infty$,

$$\|f(T_1, \ldots, T_n)\| \le \|f(B_1, \ldots, B_n)\| = \operatorname{dist}(f, J).$$

Proof: We prove first that $\Psi_{T,J}$ is WOT-continuous. Let $f_i, f \in \mathcal{F}^\infty$ with

$$\text{WOT-}\lim_i f_i(B_1, \ldots, B_n) = f(B_1, \ldots, B_n).$$

According to Lemma 4.4, we infer that WOT-$\lim_i P_{\mathcal{N}_J} f_i|_{\mathcal{N}_J} = P_{\mathcal{N}_J} f|_{\mathcal{N}_J}$. Applying Proposition 4.1, we infer that

$$w^*\text{-}\lim_i (f_i + J) = f + J. \tag{4.4}$$

For each $h, k \in \mathcal{H}$, define $\Phi(f) := \langle \Psi_T(f)h, k\rangle$. Since Ψ_T is WOT-continuous, Φ is WOT-continuous, and hence w^*-continuous. On the other hand, $\Psi(J) = 0$, so that $\Phi \in {}^\perp J$. Since (4.4) holds, we deduce that $\lim_i \Phi(f_i) = \Phi(f)$, which is equivalent to

$$\lim_i \langle f_i(T_1, \ldots, T_n)h, k\rangle = \langle f(T_1, \ldots, T_n)h, k\rangle$$

for any $h, k \in \mathcal{H}$.

According to the von Neumann inequality [Po5], for any $\psi \in J \subset \operatorname{Ker} \Psi_T$, we have

$$\|f(T_1, \ldots, T_n)\| = \|(f + \psi)(T_1, \ldots, T_n)\| \le \|f + \psi\|_\infty.$$

Using Theorem 4.1, we infer that

$$\begin{aligned}\|f(T_1, \ldots, T_n)\| &\le \operatorname{dist}(f, J) = \|P_{\mathcal{N}_J} f|_{\mathcal{N}_J}\| \\ &= \|f(B_1, \ldots, B_n)\|.\end{aligned}$$

In a similar manner, one can prove that $\Psi_{T,J}$ is a completely contractive homomorphism. This completes the proof. ∎

The following F^∞-extension is related to Example 3.3.

COROLLARY 4.6: *Let* $T = [T_1, \ldots, T_n]$ *be a c.n.c. contraction satisfying the following commutation relations:*

$$T_j T_i = \lambda_{ji} T_i T_j \quad \text{for every } 1 \le i < j \le n,$$

where $\lambda_{ij} \in \mathbb{C}$ *for* $1 \le i < j \le n$*. If* J *is the* WOT*-closed, 2-sided ideal generated by* $\{e_j \otimes e_i - \lambda_{ji} e_i \otimes e_j : 1 \le i < j \le n\}$ *in* $\mathcal{F}^\infty$*, then there is a* WOT*-continuous functional calculus given by* (4.3).

5. Representations of quotients of dual algebras

Recall that an operator algebra is a closed subalgebra of $B(\mathcal{H})$ and that a dual algebra is a unital w^*-closed subalgebra of $B(\mathcal{H})$. In the late 60's, Cole (see [BD, pages 270–273]) proved that quotients of uniform algebras are operator algebras. Shortly after, Lumer and Bernard proved that quotients of operator algebras are isometrically isomorphic to operator algebras. In [Pi, Chapter 4] Pisier noted that these methods also show that quotients of operator algebras are completely isometrically isomorphic to operator algebras. In this section we will follow these ideas closely to obtain simple representations of quotients of dual algebras. As an application, we give an alternative proof of Corollary 2.10 that does not depend on the commutant lifting theorem of [Po6].

PROPOSITION 5.1: *Let A be a unital, w^*-closed subalgebra of the bounded operators on a separable Hilbert space $\mathcal{H}$ such that for each $k \geq 1$, $M_k(A)$ has property $\mathbb{A}_1(1)$, and let J be a w^*-closed, 2-sided ideal of A. Then there exists a subspace $\mathcal{E} \subset \ell_2 \otimes \mathcal{H}$ such that the map $\widehat{\Psi} : A/J \to B(\mathcal{E})$ defined by $\widehat{\Psi}(a + J) = P_{\mathcal{E}}(I_{\ell_2} \otimes a)|_{\mathcal{E}}$ is a completely isometric representation.*

Proof: Let $x \in M_k(A)/M_k(J)$, $\|x\| = 1$. We claim that for every $\epsilon > 0$, there exists a subspace $E \subset \ell_2^k(\mathcal{H})$ such that the map

$$\Psi_x\colon A/J \to B(E) \quad \text{defined by} \quad \Psi_x(a + J) = P_E(I_{M_k} \otimes a)|_E \tag{5.1}$$

is a completely contractive homomorphism which satisfies $\|(I_{M_k} \otimes \Psi_x)(x)\| \geq 1 - \epsilon$. If we take direct sums $\bigoplus_{x,\epsilon>0} \Psi_x$, where x runs over the unit ball of $M_k(A)/M_k(J)$, $k \geq 1$, and $\epsilon > 0$, we get a completely isometric embedding of A/J. It will be clear from the construction that it is enough to take countably maps Ψ_x, so the proposition follows.

Let $\epsilon > 0$ and write $x = y + M_k(J)$, where $y = (y_{ij}) \in M_k(A)$. Find $f \in (M_k(A)/M_k(J))^* = M_k(J)^\perp$, $\|f\| = 1$, such that $\langle x, f\rangle = 1$. Since $({}^\perp M_k(J))^{**} = M_k(J)^\perp$, we can find $g \in {}^\perp M_k(J)$, $\|g\| \leq 1$, such that $|\langle y, g\rangle - \langle y, f\rangle| < \epsilon$. Then $|\langle y, g\rangle| \geq 1 - \epsilon$. Since $M_k(A)$ has the $\mathbb{A}_1(1)$ property, find $\varphi, \psi \in \ell_2^k(\mathcal{H})$, $\|\varphi\|_2 = \|\psi\|_2 = 1$ such that for each $\eta \in M_k(A)$, $\langle g, \eta\rangle = \langle \eta\varphi, \psi\rangle$.

Let $E_1 = \overline{\operatorname{span}}\{\eta\varphi : \eta \in M_k(A)\} \subset \ell_2^k(\mathcal{H})$, $E_2 = \overline{\operatorname{span}}\{\xi\varphi : \xi \in M_k(J)\} \subset E_1$, and $E = E_1 \ominus E_2$. Since E_1 and E_2 are invariant under $M_k(A)$, the map $\Phi_E\colon M_k(A) \to B(\ell_2^k(\mathcal{H}))$, defined by $\Phi_E(\eta) = P_E\eta|_E$, is a completely contractive homomorphism that vanishes on $M_k(J)$. Hence, the map

$$\Psi_x(a + J) = \Phi_E(I_{M_k} \otimes a)$$

of (5.1) is a well defined completely contractive representation.

Since $\varphi \in E_1$, $\psi \in E_2^\perp$, and $E_2^\perp$ is invariant under the adjoints of $M_k(A)$, it is easy to check that $\langle g, y\rangle = \langle y\varphi, \psi\rangle = \langle yP_E(\varphi), P_E(\psi)\rangle$. Hence,

$$\left|\langle \Phi_E(y)P_E\varphi, P_E\psi\rangle\right| = \left|\langle g, y\rangle\right| \geq 1-\epsilon.$$

For each $i, j \leq k$, let $E_{ij} = \Phi_E(e_{ij}\otimes 1_A)$. The E_{ij}'s are matrix units on E, which decompose $E = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n$. E_{ii} is the orthogonal projection onto $\mathcal{H}_i$ and E_{ij} is a partial isometry from $\mathcal{H}_j$ onto $\mathcal{H}_i$. Note that E_{ij} commutes with the range of Ψ_x, $E_{ij} = E_{ii}E_{i1}E_{1j}E_{jj}$, and $\Phi_E(e_{ij}\otimes y_{ij}) = \Phi_E((e_{ij}\otimes 1_A)(I_{M_k}\otimes y_{ij})) = \Phi_E(e_{ij}\otimes 1_A)\Phi_E(I_{M_k}\otimes y_{ij}) = E_{ij}\Psi_x(y_{ij}+J)$. Let $\varphi_j = E_{jj}P_E\varphi$ and $\psi_i = E_{ii}P_E\psi$. Then

$$\begin{aligned}\langle \Phi_E(y)P_E\varphi, P_E\psi\rangle &= \sum_{i,j\leq k}\langle \Phi_E(e_{ij}\otimes y_{ij})P_E\varphi, P_E\psi\rangle \\ &= \sum_{i,j\leq k}\langle E_{ij}\Psi_x(x_{ij})P_E\varphi, P_E\psi\rangle \\ &= \sum_{i,j\leq k}\langle \Psi_x(x_{ij})E_{1j}\varphi_j, E_{1i}\psi_i\rangle \\ &= \langle (I_{M_k}\otimes \Psi_x)(x)\hat{\varphi}, \hat{\psi}\rangle,\end{aligned}$$

where $\hat{\varphi} = (E_{11}\varphi_1, \ldots, E_{1k}\varphi_k) \in \ell_2^k(E)$, $\hat{\psi} = (E_{11}\psi_1, \ldots, E_{1k}\psi_k) \in \ell_2^k(E)$. Since $\|\hat{\varphi}\|_2^2 = \sum_{j\leq k}\|E_{ij}\varphi_j\|_2^2 \leq \sum_{j\leq k}\|\varphi_j\|_2^2 = \|P_E\varphi\|_2^2 \leq 1$, and, similarly, $\|\hat{\psi}\|_2 \leq 1$, we get

$$\left\|(I_k\otimes\Psi_x)(x)\right\| \geq \left|\langle (I_k\otimes \Psi_x)(x)\hat{\varphi}, \hat{\psi}\rangle\right| = \left|\langle \Phi_E(y)P_E\varphi, P_E\psi\rangle\right| \geq 1-\epsilon,$$

which proves the claim. Finally, notice that the map Ψ_x was determined by $g \in {}^\perp M_k(J) \subset (M_k(A))_*$. Since $(M_k(A))_*$ is separable for each $k \geq 1$, it is enough to take only countably many maps. ∎

The proof of the next corollary follows easily from the proof of Proposition 5.1. Notice that property $\mathbb{A}_1(1)$ can be used to give more explicit representations of quotient algebras than those appearing in Theorem 3.2 of [CW] and Theorem 0.3 of [Mc].

COROLLARY 5.2: *Let A be a unital, w^*-closed subalgebra of $B(\mathcal{H})$ with the $\mathbb{A}_1(1)$ property and let $J \subset A$ be a w^*-closed 2-sided ideal. Then for every $T \in A$,*

$$\operatorname{dist}(T, J) = \sup\left\{\left\|P_{E_\varphi}T|_{E_\varphi}\right\|_{B(\mathcal{H})} : \varphi \in \mathcal{H}\right\},$$

where $E_\varphi = \overline{\text{span}}\{a\varphi\colon a \in A\} \ominus \overline{\text{span}}\{b\varphi\colon b \in J\} \subset \mathcal{H}$.

Moreover, it is well known that if $A \subset B(\mathcal{H})$ is a unital w^*-closed subalgebra of $B(\mathcal{H})$, the ampliation $A^{(\infty)} = \{I_{\ell_2} \otimes a\colon a \in A\} \subset B(\ell_2 \otimes H)$ is a unital w^*-closed subalgebra of $B(\ell_2 \otimes H)$ with the $\mathbb{A}_1(1)$ property (see e.g., [Az, Section 2]). Since $M_k(A^{(\infty)}) = \big(M_k(A)\big)^{(\infty)}$, it follows that $M_k\big(A^{(\infty)}\big)$ has the $\mathbb{A}_1(1)$ property for every $k \geq 1$.

Applying Proposition 5.1 to $A^{(\infty)}$ and noticing that $I_{\ell_2} \otimes A^{(\infty)}$ is canonically isomorphic to $I_{\ell_2} \otimes A$, we obtain the following.

Corollary 5.3: *Let $A \subset B(\mathcal{H})$ be a unital, w^*-closed subalgebra of $B(\mathcal{H})$ and let $J \subset A$ be a w^*-closed 2-sided ideal. Then there exists a subspace $\mathcal{E} \subset \ell_2 \otimes \mathcal{H}$ such that the map $\widehat{\Psi}\colon A/J \to B(\mathcal{E})$ defined by $\widehat{\Psi}(a + J) = P_{\mathcal{E}}(I_{\ell_2} \otimes a)|_{\mathcal{E}}$ is a completely isometric representation.*

An alternative proof for Corollary 2.10, i.e., $\Phi\colon F^\infty/J \to P_{\mathcal{N}_J}F^\infty|_{\mathcal{N}_J}$ defined by $\Phi(f) = P_{\mathcal{N}_J} f|_{\mathcal{N}_J}$ is a completely isometric representation, can be obtained using Theorem 3.7 and Corollary 5.3 as follows.

Alternative Proof of Corollary 2.10: From Corollary 5.3 (or from Proposition 5.1 if we use that F^∞ has property $\mathbb{A}_{\aleph_0}(1)$) there exists a subspace $\mathcal{E} \subset \ell_2 \otimes \mathcal{F}^2$ such that the map $\widehat{\Psi}\colon F^\infty/J \to B(\mathcal{E})$, defined by $\widehat{\Psi}(\eta) = P_{\mathcal{E}}\eta|_{\mathcal{E}}$, is a completely isometric homomorphism. Let $\varphi \in \mathcal{E}$ and notice that $\{I_{\ell_2} \otimes S_j\colon j \leq n\}$ satisfies (2.1). Then

$$\sum_{|\alpha|=k} \|\widehat{\Psi}(S_\alpha + J)^*\varphi\|_2^2 = \sum_{|\alpha|=k} \|P_{\mathcal{E}}(I_{\ell_2} \otimes S_\alpha^*)\varphi\|_2^2 \leq \sum_{|\alpha|=k} \|(I_{\ell_2} \otimes S_\alpha^*)\varphi\|_2^2 \to 0.$$

This shows that $[\widehat{\Psi}(S_1 + J), \ldots, \widehat{\Psi}(S_n + J)]$ is C_0-contractive.

Notice that for each $\varphi \in J$, $\varphi\big(\widehat{\Psi}(S_1+J), \ldots, \widehat{\Psi}(S_n+J)\big) = \widehat{\Psi}(\varphi+J) = 0$. Then, from Theorem 3.7, there exists a unital, completely contractive, w^*-continuous map $\Phi_K\colon B(\mathcal{N}_J) \to B(\mathcal{E})$ satisfying $\Phi_K(B_\alpha) = \widehat{\Psi}(S_\alpha + J)$ for every $\alpha \in \mathbb{F}_n^+$. Recall that $B_\alpha = \Phi(S_\alpha + J)$. Hence, for each $\alpha \in \mathbb{F}_n^+$, $\widehat{\Psi}(S_\alpha + J) = \Phi_K \circ \Phi(S_\alpha + J)$. Using the w^*-continuity of the three maps, we obtain the following commutative diagram:

$$\begin{array}{ccc} F^\infty/J & \xrightarrow{\ \widehat{\Psi}\ } & B(\mathcal{E}) \\ \Phi \searrow & & \nearrow \Phi_K \\ & B(\mathcal{N}_J) & \end{array} .$$

Since Φ_K and Φ are completely contractive, and since $\widehat{\Psi}$ is completely isometric, we conclude that Φ is completely isometric. ∎

Corollary 2.10 and the following simple lemma can be used to derive Theorem 2.4. Thus, we can prove this result without using the commutant lifting theorem of [Po3]. Notice that, using a standard w^*-continuity argument, we can assume that the W_j's of Theorem 2.4 are $N \times N$ matrices. We leave the details to the reader.

LEMMA 5.4: *Let $\lambda_j = (\lambda_{j1}, \ldots, \lambda_{jn}) \in \mathbb{B}_n$, $j = 1, \ldots, k$, be k different points in $\mathbb{B}_n$. For each $i_0 \in \{1, \ldots, k\}$ there exists $\varphi_{i_0} \in F^\infty$ such that $\varphi_{i_0}(\lambda_{i_0}) = 1$ and $\varphi_{i_0}(\lambda_j) = 0$ whenever $i_0 \neq j$. Consequently, given $W_1, \ldots, W_k \in M_N$, there exists $\varphi \in M_N(F^\infty)$ such that $\varphi(\lambda_j) = W_j$ for every $j = 1, \ldots, k$.*

Proof: Fix $i_0 \in \{1, \ldots, k\}$. For each $j \neq i_0$ find $q \in \{1, \ldots, n\}$ such that $\lambda_{i_0 q} \neq \lambda_{jq}$ and define $\theta_j = S_q - \lambda_{jq} I$. Then $\theta_j(\lambda_{i_0}) = \lambda_{i_0 q} - \lambda_{jq} \neq 0$ and $\theta_j(\lambda_j) = 0$. Let $\psi = \otimes_{j \neq i_0} \theta_j$. Then $\psi(\lambda_{i_0}) \neq 0$ and $\psi(\lambda_j) = 0$ whenever $j \neq i_0$. Define $\varphi_{i_0} = \frac{1}{\psi(\lambda_{i_0})} \psi$. If $W_1, \ldots, W_k \in M_N$, then $\varphi = \sum_{i \leq k} W_i \otimes \varphi_i \in M_N(F^\infty)$ satisfies $\varphi(\lambda_i) = W_i$ for $i \leq k$. ∎

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